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Abstract

A major difficulty in traditional finite element analysis is the effort of meshing complex three-dimensional solids and structures. The inherent reason for the difficulty is that highly distorted elements should be avoided. This condition is difficult to satisfy because the traditional elements must abut to each other, that is, they cannot overlap. To overcome this restriction, we have developed "overlapping elements". These elements perform well even when highly distorted and hence can be used much more easily in meshing a complex domain. However, they use additional nodal degrees of freedom which add to the computational effort of solution. To reduce the overall solution cost, including the meshing, we focus on the AMORE scheme in which traditional undistorted elements are used to discretize most of the analysis domain and overlapping elements are used for the rest of the domain. The premise is that in this way, the meshing effort is much reduced and the computational effort is also less than in a traditional finite element analysis. This way the use of AMORE leads to an overall efficient modern finite element analysis.

Keywords: finite elements, overlapping finite elements, AMORE, automatic meshing with overlapping and regular elements, statics and dynamics, computational effort

1 Introduction

The analysis of a complex three-dimensional solid or fluid requires a significant effort for meshing the analysis domain. The underlying reason for this effort is that the finite elements need to abut to each other, without gaps, and the elements should not be highly distorted [1]. For this reason, much research effort has been focused on meshless methods of analysis in which no underlying mesh is needed [2].

One of the computationally most effective meshless techniques is the Method of Finite Spheres [3]. In this analysis procedure small and large spheres can be used, with no restriction, the spheres overlap, no artificial analysis factors for stability are used and the procedure is general to solve two- and three-dimensional problems in statics and dynamics [4,5].

However, while this method is one of the computationally most effective meshless procedures, and the meshing can be relatively easily performed because "no spheres/elements need to abut", the computational effort to establish the governing equations and for the solution of these equations is very high, too high for the method to be a competitive analysis tool in industry. Therefore, it appears that while numerous papers by many authors have been written on meshless methods, these techniques will not have the desired impact in practical engineering analyses – at least not for now, with the current hardware and software available.

To tackle the basic problem of reducing the meshing effort while at the same time arriving at a computationally efficient procedure, it appears natural to therefore combine the "best qualities" of the schemes used in traditional finite element analysis and the more recently developed meshless methods. This thought led to the development and use of 'overlapping finite elements' and the AMORE paradigm [6-9]. Here AMORE is an acronym for **A**utomatic **M**eshing with **O**verlapping and **R**egular **E**lements [9]. Instead of overlapping elements, also 'overlapping meshes' can be used [10] but we focus here on overlapping elements.

In the next sections we give a short overview of the formulations of overlapping elements and the AMORE paradigm and point out the advantages in use. We also give a few illustrative solutions. However, for more depth regarding the formulations and applications, the interested reader is referred to the earlier published papers. We conclude this exposition with a view towards further valuable developments of the overlapping elements and AMORE.

2 The Overlapping Elements and AMORE

In this section we briefly overview the formulations of the overlapping elements and the AMORE scheme.

2.1 Overlapping elements

Consider the part of a mesh shown in Figure 1, which shows three two-dimensional polygonal elements that overlap. The polygonal elements are colored in blue, green and red and overlap over the triangular region $i - j - k$. We focus on the behavior of this triangular region, for example a displacement component u is assumed to be:

$$u(\mathbf{x}) = \sum_{I=1}^{N} h_I \psi_I \tag{1}$$

where h_I is the traditional low-order finite element shape function (in Figure 1, $N = 3$), the three-node triangular region corresponds to the nodes $i = 1$, $j = 2$, $k = 3$, and ψ_I is the field "corresponding to the polygonal element I associated with node I", see Figure 1. For ψ_I, we use

$$\psi_I = \sum_{K=1}^{m} \phi_K^I \, u_K \tag{2}$$

where m is the number of nodes of the polygonal element, $m = 6$ or 7 in Figure 1, and u_K is the unknown nodal basis function for node K. In static analysis, u_K corresponds to a polynomial basis [7,8], but it can be efficient, for example, to include harmonic functions in the analysis of wave propagations [11,12].

Since we use Eq. (2) in Eq. (1) with h_I (and its properties) we can use N instead of m and arrive at

$$u(\mathbf{x}) = \sum_{K=1}^{N} \rho_K \, u_K \tag{3}$$

where

$$\rho_K = \sum_{I=1}^{N} h_I \, \phi_K^I \tag{4}$$

For the triangular overlapping element considered in Figure 1 we have $N = 3$ but the formulation is very general. We have formulated in this way, two-dimensional triangular and quadrilateral elements, and three-dimensional tetrahedral, brick, pyramid, and prism elements [7-9,13-15].

An important ingredient in the element formulations is the choice of ϕ_K^I which is selected for stability, to satisfy the patch test, for which we need to have $\sum_{K=1}^{N} \phi_K^I = 1$ for all I, and optimal accuracy. Additional important ingredients are the coupling elements which are used to couple regular elements to the overlapping elements. We refer to the references (e.g. Refs. [9,14,15]) for the construction of effective ϕ_K^I and the coupling elements.

The overlapping finite elements have the following important properties: they are compatible and satisfy the patch test, and they are quite insensitive to element geometric distortions. As an example, the distortion insensitivity compared to the use of

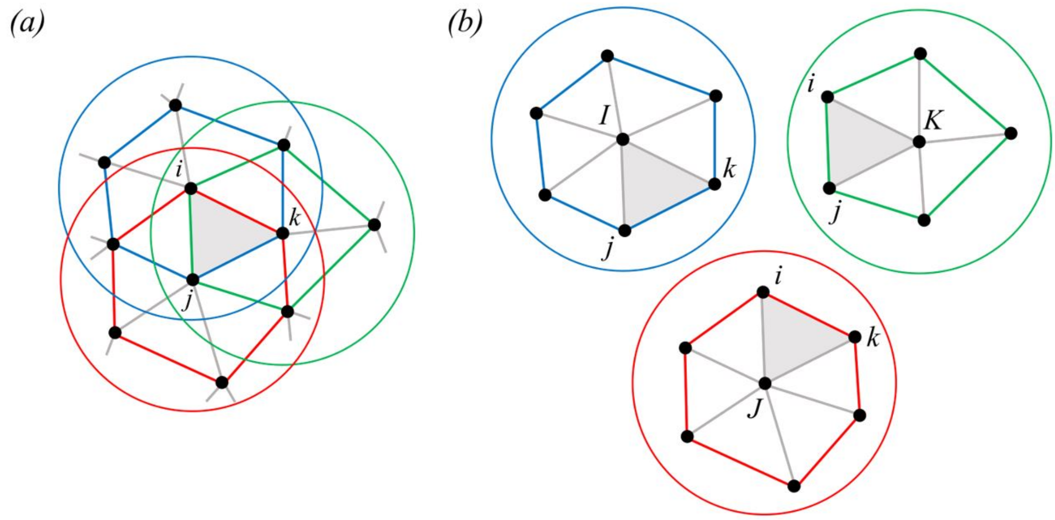

Figure 1: Schematic showing three polygonal elements used to construct the triangular 3-node overlapping element; (a) the triangular element $i - j - k$; (b) the corresponding polygonal elements with 6 or 7 nodes and center nodes I, J, K. The schematic also shows the spheres underlying the formulation of the element.

traditional elements is displayed in the example solution of Figure 2 and Table 1. Here we compare the calculated response using a 4-node overlapping finite element (with a quadratic nodal polynomial) to the response predicted using the traditional 9-node element [14]. The 9-node element is quite powerful because it preserves its completeness when subjected to angular geometric distortions [1].

The further potential of the overlapping elements is displayed in Figures 3 and 4, where we show the use of overlapping elements in the solution of a wave propagation problem. We consider the analysis of Lamb's problem subjected to step loadings [12] given by $F_c(t)$

$$F_c(t) = 2 \times 10^6 [H(0.15 - t) - 3H(0.1 - t) + 3H(0.05 - t)] \tag{5}$$

where H is the unit step function. The discontinuous loading causes high frequency oscillations and hence for this response prediction we included harmonic functions in the basis of the overlapping elements. To model the infinite domain, we used a uniform mesh of $160 \times 160 \times 2 = 51,200$ triangular (3-node) elements and a non-uniform mesh of $30,446$ triangular elements, each time for half of the domain using symmetry. For the time integration we employed the Bathe implicit time integration scheme [12,16,17].

Figure 4 shows the calculated response in comparison with an analytical solution [12]. We see that the predicted response is excellent with both meshes used. More details of this solution and other wave propagation analyses using the overlapping el-

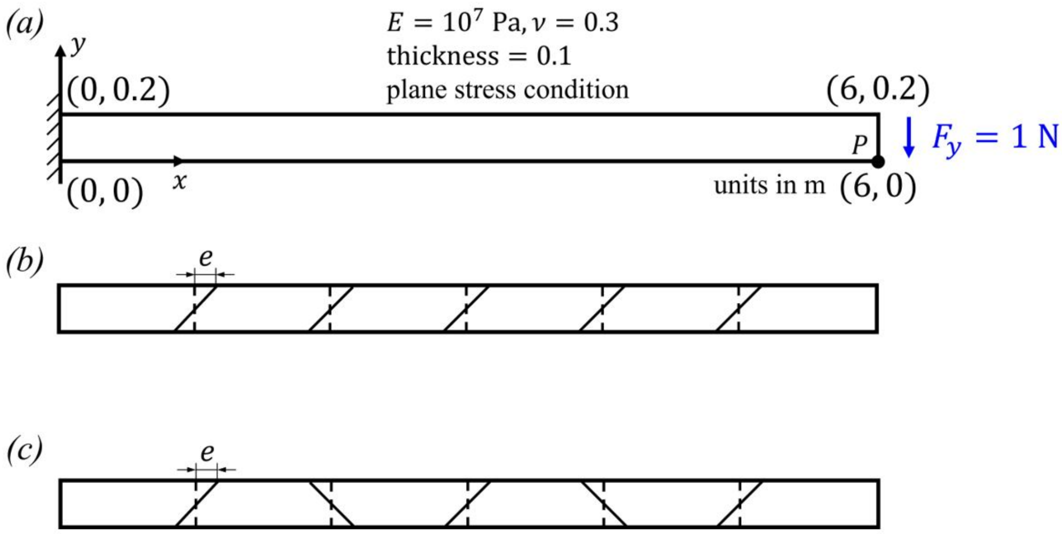

Figure 2: Thin beam problem; (a) Description of the bending problem; total applied force is 1N; (b) Parallelogram mesh used; (c) Trapezoidal mesh used [14].

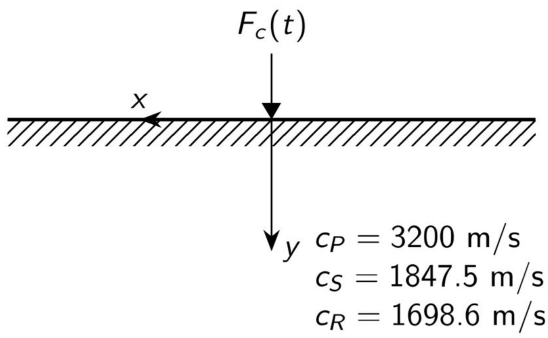

Figure 3: A semi-infinite elastic medium in plane strain conditions subjected to a concentrated line force on the free surface; c_P, c_S, c_R are the pressure, shear and Rayleigh wave speeds.

ements are given in Refs. [11,12]. Of course, to solve wave propagation problems not only the spatial discretization of the domain is important but also the temporal discretization, that is, the time integration scheme. These two procedures for the spatial and the temporal solution need 'to work together' and influence each other such as to reach an accurate prediction of the response. The overlapping elements have shown good characteristics in that regard, see Refs. [11,12].

While the overlapping elements clearly show specific strengths, a disadvantage of the elements is that they carry additional nodal degrees of freedom, since u_K corresponds to a polynomial (or another suitable function). Hence the elements require a

9 node FE (72 dofs)	$e = 0m$	0.1	0.2	0.3	0.4
parallelogram	0.9901	0.9813	0.9397	0.8770	0.8252
trapezoidal		0.9811	0.9234	0.8422	0.7966
4-node OFE (156 dofs)	0	0.1	0.2	0.3	0.4
parallelogram	0.9909	0.9917	0.9925	0.9920	0.9905
trapezoidal		0.9913	0.9910	0.9903	0.9906

Table 1: Normalized y-direction displacement at the tip ($x = 6$, $y = 0$); for the normalization the reference displacement is -0.1081m [14].

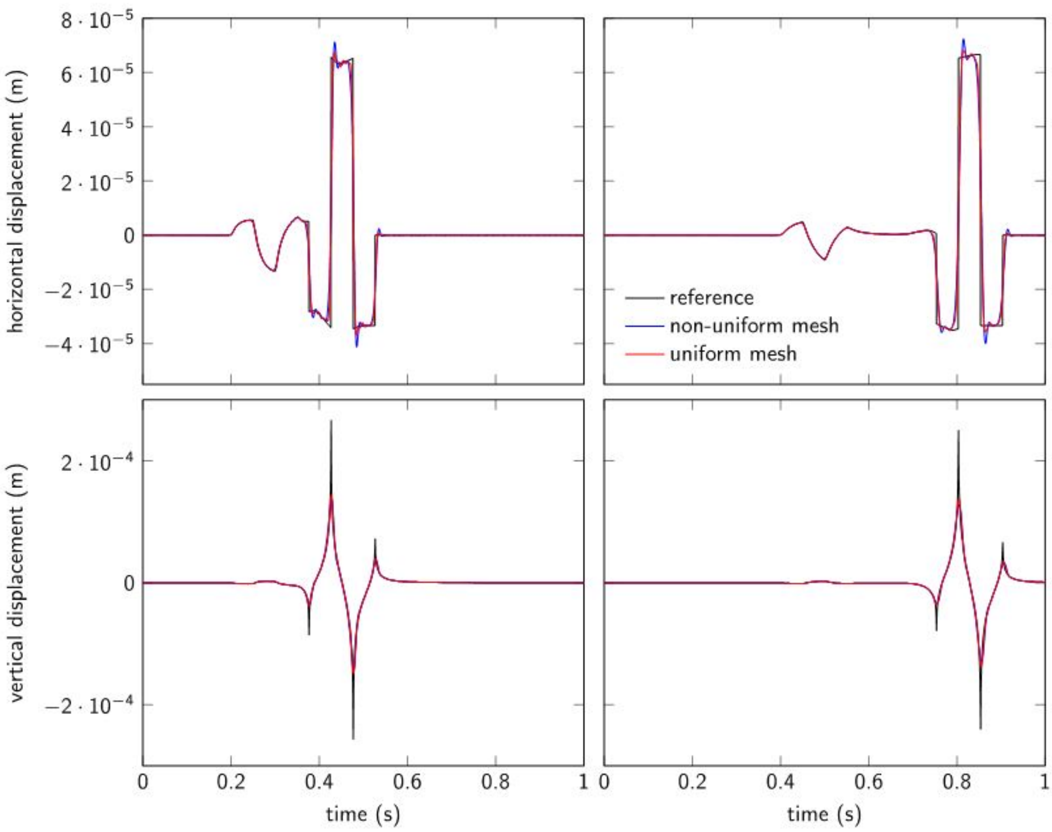

Figure 4: Time history of horizontal and vertical displacements of the elastic medium subjected to the step loading on the free surface. The displacements at $\mathbf{x} = (640, 0)$ and $\mathbf{x} = (1280, 0)$ are shown.

higher order numerical integration in the evaluation of the stiffness and mass matrices than the traditional elements with the same number of nodes, and the bandwidth is larger in the assembled system matrices.

To obtain insight, it is of value to see the relationship of the overlapping finite elements to traditional finite elements and 'finite elements with interpolation covers' [18,19]. The finite elements with covers are obtained by neglecting the 'displacement bubbles' implicitly introduced in the overlapping elements, and the traditional finite elements are obtained by in addition only assigning the usual (traditional) nodal degree of freedom at each node. Hence the overlapping finite elements can be thought of as 'containing' the finite elements with interpolation covers and the traditional finite elements, or the overlapping elements may be regarded as further developments of these two discretization schemes.

Finally, we should also mention that the work reported upon in this paper relates closely to the important contributions of other researchers, in particular to the development of the 'generalized finite element method' [20-24]. However, the overlapping finite elements are simple in formulation and stable, without special considerations. Elements for one-, two- and three-dimensional analyses have been formulated with smooth stresses inside each element. Also, as expected, the overlapping elements are implemented and employed with coupling elements like traditional elements in a finite element program and can be used in one mesh with regular elements.

The effective use of the elements is reached in the AMORE paradigm of analysis which we consider next.

2.2 The AMORE scheme

The AMORE procedure is based on the knowledge that:

1. Traditional finite elements can be quite effective when they are geometrically undistorted and are formulated with incompatible modes. A typical such element is the 4-node undistorted quadrilateral element with incompatible modes for two-dimensional solutions and the corresponding 8-node brick element with incompatible modes for three-dimensional analyses.

2. The overlapping elements are quite distortion-insensitive, that is, their predictive capability is not significantly diminished when the elements are geometrically distorted. However, the elements are computationally more expensive in use than traditional elements due to the use of a higher order numerical integration for the calculation of the governing matrices and the increased number of degrees of freedom at their nodes.

Hence it is natural to mesh a domain as much as possible with undistorted regular elements, and mesh the rest of the domain with overlapping elements. This aim is pursued in the AMORE scheme.

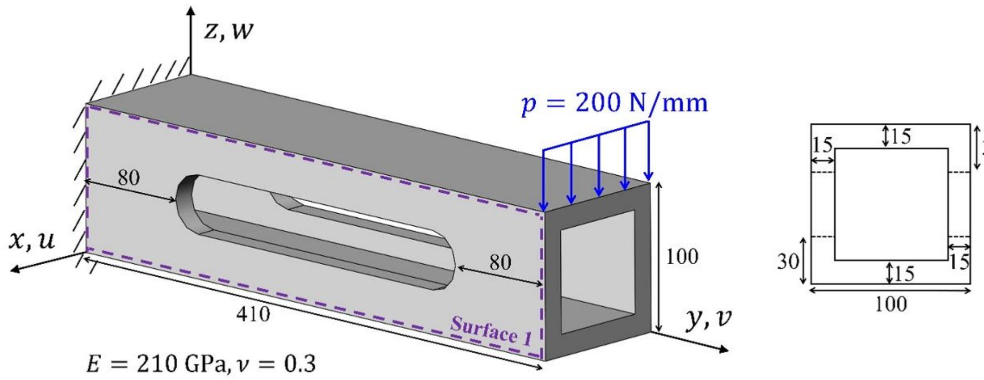

Figure 5: Description of machine tool jig problem; a uniformly distributed load p per unit length is applied, lengths in mm.

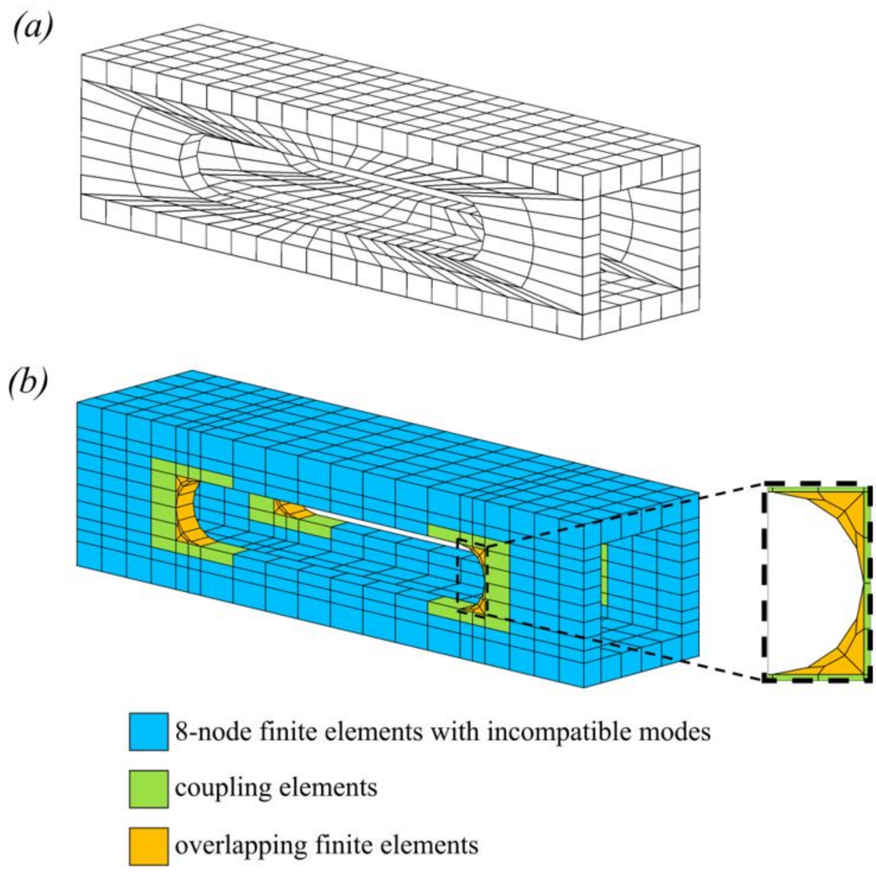

Figure 6: Meshes used for solving the machine tool jig problem; (a) Traditional mesh using the 8-node brick element with incompatible modes; (b) AMORE mesh also including overlapping elements [14].

Figures 5 to 7 give the analysis of a machine tool jig. The reference solution on which the errors are measured was obtained using a very fine mesh of 27-node elements. We see that using the AMORE mesh gives a significantly more accurate stress prediction than using the mesh of 8-node brick elements with incompatible modes [14].

Figures 8 to 10 show the analysis results of a machine part. Here a comparison is given between the results obtained using the AMORE scheme and traditional analyses using 10-node tetrahedral elements and 27-node brick elements. The structure is clamped on its footing and a pressure loading is applied in part of the hole at its top for the structure to bend. We see that in this comparison the AMORE solution is quite competitive, however, the meshes were not optimized and the solution accuracy needs to be assessed against a very accurate solution. A more encompassing conclusion regarding the effectiveness of AMORE analyses cannot yet be made.

Indeed, for a thorough comparison of solution effectiveness, each method should be used in its optimal manner to obtain analysis results to a prescribed accuracy (in displacement, stress or a norm) and the computational effort should be measured. This comparison would then include only the computer time and storage used. At least equally important is of course also the effort needed for meshing, that is, to establish the numerical model to be solved. The premise is that then in the 'overall effort of solution' the AMORE scheme would be seen as very effective.

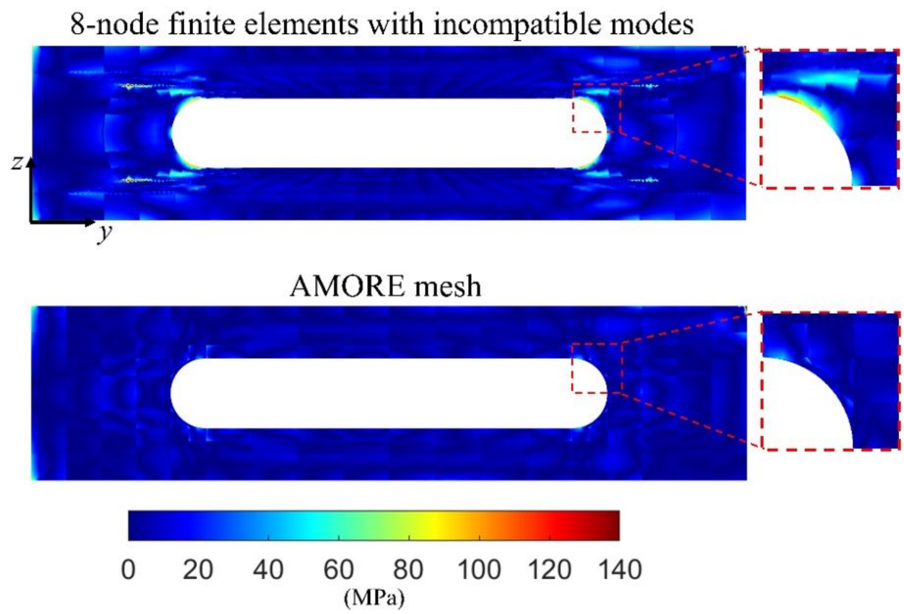

Figure 7: Errors in the von Mises stress results on surface 1 of the jig (see Figure 5). The maximum von Mises stress on the surface is about 250 MPA.

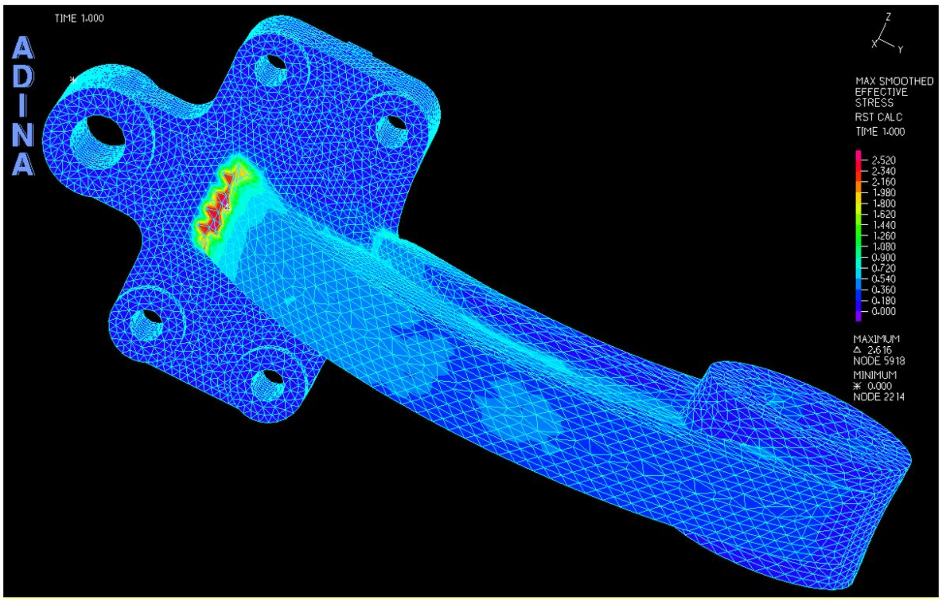

Figure 8: AMORE analysis using 8-node brick elements with incompatible modes and 4-node overlapping tetrahedral elements. Number of equations = $122,730$; max. y-displacement -22.25; max. von Mises stress = 2.62; solution time 7 sec.

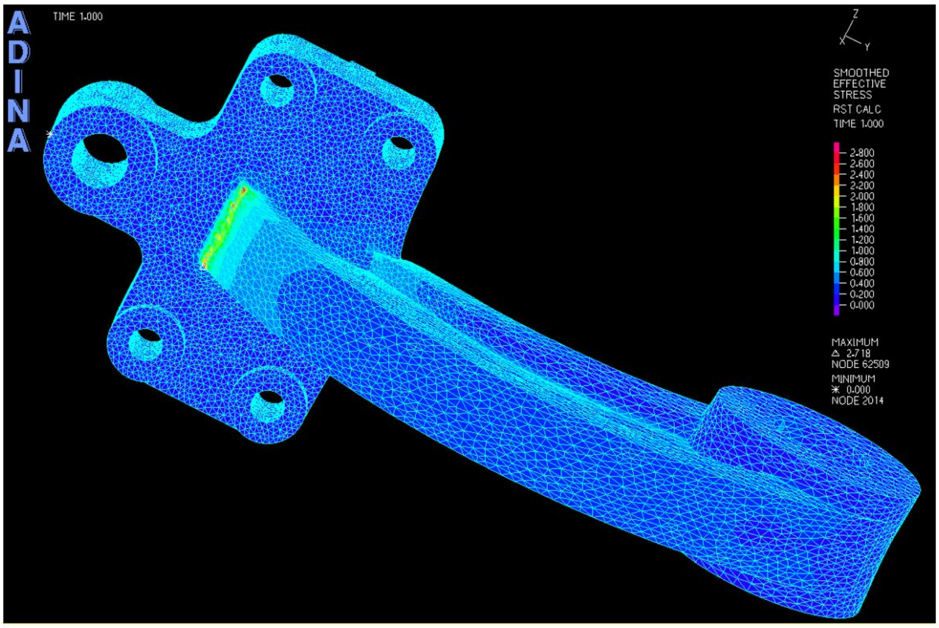

Figure 9: A traditional analysis using 10-node tetrahedral elements. Number of equations = $739,635$; max. y-displacement -22.42; max. von Mises stress = 2.72; solution time 60 sec.

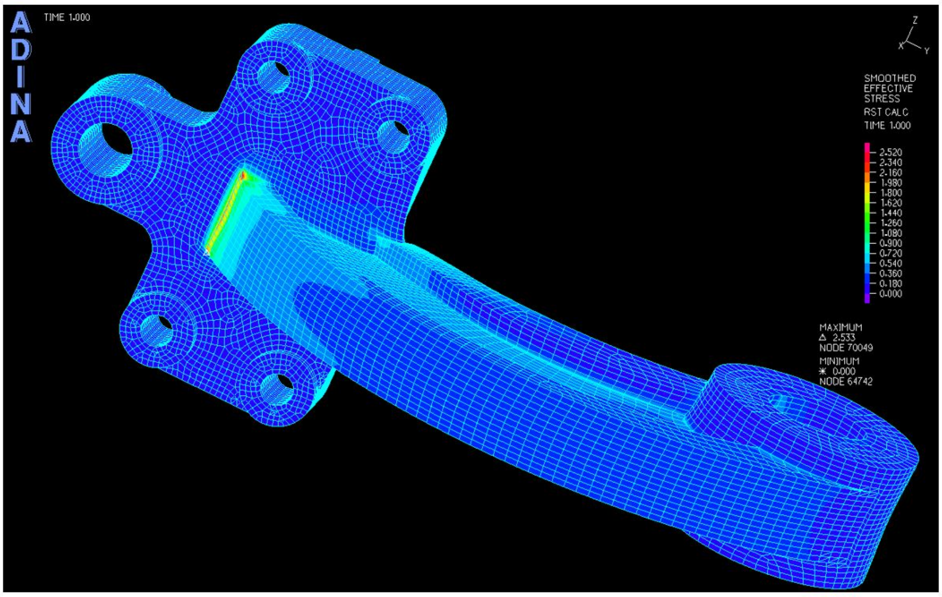

Figure 10: A traditional analysis using 27-node hexahedral elements. Number of equations = $1,197,816$; max. y-displacement -22.48; max. von Mises stress = 2.53; solution time 242 sec.

3 Concluding remarks

The objective was to give an overview of the recently developed overlapping finite elements and the AMORE scheme.

In essence, these procedures are built on using the "best" qualities of the traditional finite elements and meshless methods. However, as frequently for novel numerical schemes, further developments are needed. These include the extension of the formulations to general dynamic and nonlinear analyses, including the solution of contact conditions. Also, while the AMORE scheme is already valuable in the analysis of many problems, the effectiveness might be increased by more efficient numerical integration schemes for the overlapping element stiffness and mass matrices, optimally "adjusting" direct sparse solvers for the increased bandwidth due to the additional degrees of freedom, and the use of novel special degrees of freedom in solving certain problems in engineering and the sciences.

Indeed, also novel overlapping elements or schemes close thereto, building on the use of the "best" qualities of the traditional finite elements and meshless methods – thus still using an AMORE procedure – might be envisaged.

Finally, an effective automatic meshing of complex three-dimensional geometries focused on an efficient use of AMORE is particularly important. The mesh should take maximum advantage of using traditional undistorted elements where possible and effective, and the overlapping elements in the remaining parts of the domain. Ideally such AMORE meshes would be very simple for an analyst to generate.

With the above aims achieved, AMORE could also be a very powerful analysis tool for use by designers to be used in studies of preliminary designs where a lower but still acceptable level of accuracy in displacements and stresses would be employed, thus allowing a computationally fast solution.

Hence, there are exciting avenues to be pursued in the further development of AMORE to reach an ever more powerful analysis tool in engineering and the sciences.

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