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An exact stiffness matrix for buckling analysis of an axial-flexural coupled column including shear deformation

J.R. Banerjee

School of Science and Technology
City, University of London, Northampton Square
London, United Kingdom

Abstract

Flexural-torsional buckling analysis of columns is widely covered in the literature. By contrast, there are hardly any corresponding publications on axial-flexural buckling. This is because there appears to be a commonly held view that in a column, the axial deformation is perceived to be small and therefore its coupling with the flexure can be neglected when predicting its critical buckling load. This paper counters this view by focusing on the buckling behaviour of axial-flexural coupled columns. The usefulness for this research stems from the fact that there are many practical columns which have cross-sections that exhibit axial-flexural coupling as opposed to flexural-torsional coupling, and thus, the axial-flexural coupling is likely to have significant effects on buckling behaviour. The problem does not appear to have been adequately addressed by investigators. Starting from the derivation of the governing differential equation with the inclusion of shear deformation, the stiffness matrix of an axial-flexural coupled column is derived in an exact sense and subsequently applied through the implementation of the Wittrick-Williams algorithm as solution technique to determine the critical buckling load of the column for various boundary conditions. The results are validated by an alternative exact method. Finally. some conclusions are drawn.

Keywords: column, axial-flexural buckling, exact stiffness matrix, critical buckling load, Wittrick-Williams algorithm

1 Introduction

The literature is inundated with research papers on flexural-torsional buckling of columns [1-8]. These publications are predominantly based on the premise that the axial deformation arising from the application of a compressive load in a column is somehow negligible, suggesting that any coupling arising from the axial deformation of the column is insignificant or unimportant. A similar viewpoint prevailed amongst vibration researchers who studied the free vibration behaviour of flexure-torsion coupled beams [9-13], until recently when beams displaying axial-flexural coupling as opposed to flexural-torsional coupling were investigated [14-15] for their free vibration characteristics. These latter investigations have shown that axial-flexural coupling can have profound effects on the free vibration behaviour of beams. A review of buckling literature indicates that any possible consequences of axial-flexural coupling effect on the critical buckling load of a column have not been given sufficient importance. This is rather surprising, and this paper addresses this issue by developing an exact stiffness matrix for an axial-flexural coupled column with the inclusion of shear deformation to examine its buckling behaviour.

The principle of virtual work is invoked to derive the governing differential equations for an axial-flexural coupled column including the effect of shear deformation. The solutions of the governing differential equations yielded the displacement vector comprising the expressions for axial and flexural displacements and the cross-sectional rotation. Next, the corresponding force vector comprising the expressions for the axial force, shear force and bending moments was obtained. The force vector and displacement vector are finally related via the stiffness matrix. The ensuing stiffness matrix is used to determine the critical buckling load of an axial-flexural coupled column for various boundary conditions.

Within the above pretext, it is worth noting that there are many published works [16-24] on the flexural buckling of columns with the inclusion of shear deformation, but without considering the effect of axial-flexural coupling. These publications reveal that shear deformation can have significant effect on the critical buckling load of columns, particularly with smaller slenderness ratios. Engesser [16, 17] was probably the earliest investigator who published the effect of shear deformation on the critical buckling of a column more than a century ago. His work was given due recognition by Timoshenko in his well-publicised text [18]. Engesser's research was apparently overlooked for many years, until relatively recently when the interest in shear deformable columns resurfaced [19-24]. These latter investigations are without doubt significant and noteworthy, but they focus on the flexural buckling of shear deformable columns without the effect of the coupling between the axial and flexural deformations. The current paper addresses this issue and demonstrates the individual and combined effects of shear deformation and axial-flexural coupling on the critical buckling of columns. This is achieved by developing an exact stiffness matrix using linear small deflection theory. The developed stiffness matrix is applied by implementing the Wittrick-Williams algorithm [25] as solution technique. The results from the theory are validated by an alternative method which allows eccentrically connected columns at nodes to simulate axial-flexural coupling in an exact manner.

2 Theory

In what follows, an exact stiffness matrix of an axial-flexural coupled column with the inclusion of shear deformation is developed by using linear small deflection theory. First, the governing differential equations are derived using the principle of virtual work. Next, exact solutions for the displacements and forces are obtained in closed analytical form. Finally, the stiffness matrix is obtained by imposing the boundary conditions, and thus linking the force-displacement relationship.

2.1 Derivation of the governing differential equations and solution

In a right-handed Cartesian coordinate system, Figure 1 shows a uniform column of length L with its flexural axis which is the locus of shear centre of the cross-sections, coinciding with the Y-axis. A compressive axial force P is acting through the shear centre and hence, along the flexural axis of the column, as shown. Note that P can be negative and thus, tension is included. The coupling between axial and flexural displacements for such a column will occur because of the eccentricity of the centroid (G_c) and shear centre (E_s) of the cross-section. There are many practical cross-sections for which the centroid and shear centre are non-coincident (see Figure 2 of [14]), but the inverted T section is shown in Figure 1 only for convenience. The centroidal axis and the flexural axis of the column which are respectively the loci of the centroid and shear centre of the column cross-section are separated by a distance z_α, as shown.

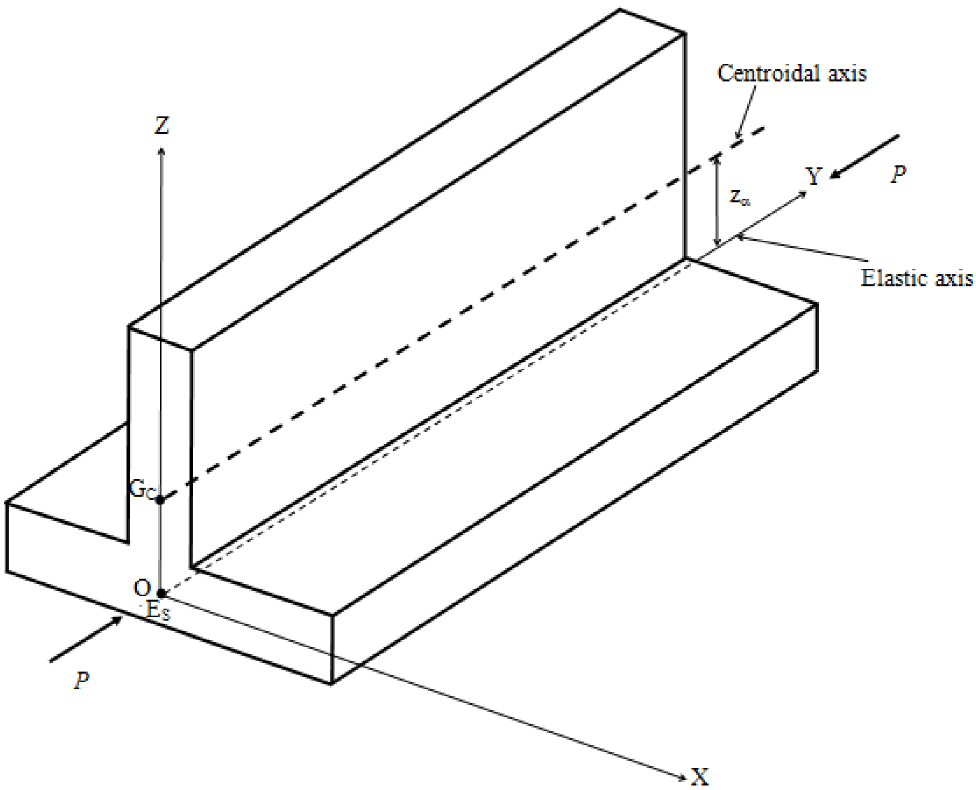

Figure 1. Coordinate system and notation for an axial-flexural coupled column.

If v, w and θ are axial displacement, flexural displacement and bending rotation of a point at a distance y from the origin and at a height z from the flexural axis, i.e., the point (y, z) in the coordinate system (see Figure 1), one can write

$$v = v_0 - z\theta \quad , \qquad w = w_0 \tag{1}$$

where v_0 and w_0 are the corresponding displacement components of the point (y, 0) on the Y-axis (i.e., the flexural axis).

Using linear, small deflection elasticity theory, the expression for the normal strain ε_y and shearing strain (γ_{yz}) can be expressed as [15,]

$$\varepsilon_y = v_0' - z\theta', \qquad \gamma_{yz} = w_0{}' - \theta \tag{2}$$

where a prime denotes differentiation with respect to y.

The potential or strain energy U_1 of the beam resulting from normal and shear strains is given by

$$U_1 = \frac{1}{2}\int_0^L \int_A E\varepsilon_y^2 \, dAdy + \frac{1}{2}\int_0^L \int_A kG\gamma_{xy}^2 \, dAdy \tag{3}$$

where E and G are the Young's modulus and modulus of rigidity (shear modulus) of the column material, respectively, k is the shear correction or shape factor, and the integrations are carried out over the cross-sectional area A and length L of the column.

Substituting ε_y and γ_{yz} from Equation (2) into Equation (3), and integrating over the uniform beam cross-section, we obtain

$$U_1 = \frac{1}{2}\int_0^L \{EA(v_0')^2 - 2EAz_\alpha v_0'\theta' + EI_e(\theta')^2 + kAG(w_0' - \theta)^2\} \, dy \tag{4}$$

where A and I_e are the area of cross-section and second moment of area about the flexural axis so that EA and EI_e are the extensional and flexural stiffnesses of the column, respectively.

The potential energy due to the externally applied compressive force P (see Figure 1) is given by

$$U_2 = -\frac{1}{2}P\int_0^L (w_0')^2 dy \tag{5}$$

Thus, the total potential energy $U = U_1 + U_2$ is given by

$$U = \frac{1}{2}\int_0^L \left\{EA(v_0')^2 - 2EAz_\alpha v_0'\theta' + EI_e(\theta')^2 + kAG(w_0' - \theta)^2 - P{w_0'}^2\right\} dy \tag{6}$$

By the postulate of the principle of virtual work, a conservative system such as the axial-flexural coupled column shown in Figure 1, will be in equilibrium if and only if the total potential energy of the system is stationary. That is

$$\delta U = 0 \tag{7}$$

Substitution of Equation (6) into Equation (7) gives

$$\int_0^L \left\{\begin{matrix} EAv_0'\delta v_0' - EAz_\alpha v_0'\delta\theta' - EAz_\alpha\theta'\delta v_0' + EI_e\theta'\delta\theta' + kAG(w_0' - \theta)\delta w_0' \\ -kAG(w_0' - \theta)\delta\theta - Pw_0'\delta w_0' \end{matrix}\right\} dy = 0 \tag{8}$$

Carrying out the integration by parts leads to the following governing differential equations and expressions for axial force f, shear force s, and bending moment m.

Governing differential equations:

$$EAv_0'' - EAz_\alpha\theta'' = 0 \tag{9}$$

$$EI_e\theta'' - EAz_\alpha v_0'' + kAG(w_0' - \theta) = 0 \tag{10}$$

$$kAG(w_0'' - \theta') - Pw_0'' = 0 \tag{11}$$

$$\textit{Axial force}: \qquad f = -EAv_0' + EAz_\alpha\theta' \tag{12}$$

$$\textit{Shear force}: \qquad s = -kAG(w_0' - \theta) + Pw_0' \tag{13}$$

$$\textit{Bending moment}: \qquad m = EAz_\alpha v_0' - EI_e\theta' \tag{14}$$

Introducing the non-dimensional length $\xi = y/L$ and then eliminating the v_0 and θ terms from Equations (9)-(11) gives the following differential equation in w_0.

$$D^2\{D^2 + \lambda^2\}w_0 = 0 \tag{15}$$

where

$$D = \frac{d}{d\xi}; \qquad \lambda^2 = \frac{p^2}{(1 - p^2s^2)(1 - \frac{\mu^2}{r^2})} \tag{16}$$

with

$$p^2 = \frac{PL^2}{EI_e}; \quad s^2 = \frac{EI_e}{kAGL^2}; \quad \mu^2 = \frac{z_\alpha^2}{L^2}; \quad r^2 = \frac{I_e}{AL^2} \tag{17}$$

The solution of the governing differential equation, i.e., Equation (15) is given by

$$w_0(\xi) = C_1 + C_2\xi + C_3\cos\lambda\xi + C_4\sin\lambda\xi \tag{18}$$

where $C_1 - C_4$ are arbitrary constants of integration.

It can be shown with the help of Equations (9)-(11) that the axial displacement v_0 and bending rotation θ are given by

$$v_0(\xi) = \mu C_2 - C_3\mu\lambda(1 - p^2s^2)\sin\lambda\xi + C_4\,\mu\lambda(1 - p^2s^2)\cos\lambda\xi + C_5 + C_6\xi \tag{19}$$

$$\begin{aligned}\theta(\xi) &= \frac{1}{L}\left(\frac{p^2s^2}{\lambda^2}D^3w_0 + Dw_0\right)\\ &= \frac{1}{L}\{C_2 - C_3\lambda(1 - p^2s^2)\sin\lambda\xi + C_4\lambda(1 - p^2s^2)\cos\lambda\xi\}\end{aligned} \tag{20}$$

where C_5 and C_6 in Equation (19) are two different additional constants.

With the help of Equations (12)-(14) and Equations (18)-(20), the expressions for axial force (*f*), shear force (*s*) and bending moment (*m*) can be written as

$$f(\xi) = -\frac{EA}{L}\left(\frac{dv_0}{d\xi} - z_\alpha\frac{d\theta}{d\xi}\right) = -\frac{EA}{L}C_6 \tag{21}$$

$$s(\xi) = \frac{EI_e}{L^3}\left(-\frac{1}{s^2}\frac{dw_0}{d\xi} + \frac{1}{s^2}\theta L + p^2\frac{dw_0}{d\xi}\right) = \frac{EI_e}{L^3}(p^2C_2 + q^2C_3\sin\lambda\xi - q^2C_4\cos\lambda\xi \tag{22}$$

$$m(\xi) = -\frac{EI_e}{L^2}\left(L\frac{d\theta}{d\xi} - \frac{\mu}{r^2}\frac{dv_0}{d\xi}\right) = \frac{EI_e}{L^2}\left(C_6\frac{\mu}{r^2} + C_3\tau^2\cos\lambda\xi + C_4\,\tau^2\sin\lambda\xi\right) \tag{23}$$

where

$$q^2 = \frac{\lambda + \psi - \lambda p^2s^2}{s^2}; \qquad \tau^2 = \lambda\psi(1 - \frac{\mu^2}{r^2}) \tag{24}$$

with

$$\psi = -\lambda(1 - p^2s^2) \tag{25}$$

2.2 Derivation of the stiffness matrix

The expressions for the axial displacement (v_0), bending displacement (w_0) and bending rotation (θ) together with the expressions for axial force (f), shear force (s) and bending moment (m) given above can now be used to derive the stiffness matrix of the coupled axial-flexural column by applying the boundary conditions. Referring to the sign convention for positive axial force, shear force and bending moment shown in Figure 2, the following boundary conditions for displacements and forces as shown in Figure 3 are applied.

$$\text{At } \xi = 0\text{: } v_0 = V_1;\ w_0 = W_1;\ \theta = \Theta_1;\ f = F_1;\ s = S_1;\ m = M_1 \quad (26)$$

$$\text{At } \xi = 1\text{: } v_0 = V_2;\ w_0 = W_2;\ \Theta = \Theta_2;\ f = -F_2;\ s = -S_2; m = -M_2 \quad (27)$$

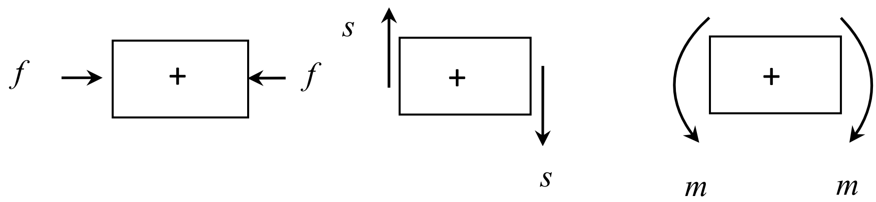

Figure 2. Sign convention for axial force *f*, shear force *s* and bending moment *m*.

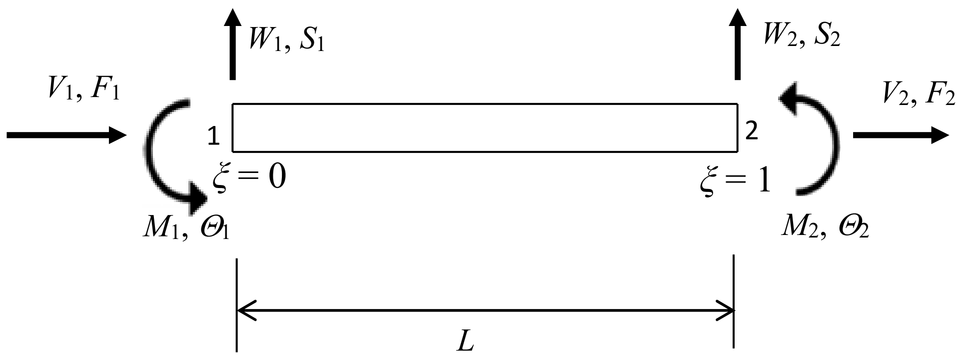

Figure 3. Boundary condition for displacements and forces for a coupled axial-flexural column.

The displacement vector $\boldsymbol{\delta}$ and the force vector $\mathbf{F}$ of the column connecting the ends (nodes) 1 and 2, see Figure 3, can be expressed as:

$$\boldsymbol{\delta} = [V_1\ W_1\ \Theta_1\ V_2\ W_2\ \Theta_2]^T; \quad \mathbf{P} = [F_1\ S_1\ M_1\ F_2\ S_2\ M_2]^T \quad (28)$$

where the upper suffix T denotes a transpose.

The displacement vector $\boldsymbol{\delta}$ and the constant vector $\mathbf{C}$ (with C_i, i = 1,2, …6) can now be related using Equations (18)-(20) and Equations (26)-(27) to give

$$\boldsymbol{\delta} = \mathbf{A}\,\mathbf{C} \quad (29)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & \mu & 0 & -\mu\psi & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1/L & 0 & -\psi/L & 0 & 0 \\ 0 & \mu & \mu\psi\sin\lambda & -\mu\psi\cos\lambda & 1 & 1 \\ 1 & 1 & \cos\lambda & \sin\lambda & 0 & 0 \\ 0 & 1/L & \psi\sin\lambda/L & -\psi\cos\lambda/L & 0 & 0 \end{bmatrix} \tag{30}$$

In a similar manner, the relationship between the force vector **F** and the constant vector **C** is established by using Equations (21)-(23) and Equations (26)-(27) to give

$$\mathbf{F} = \mathbf{B}\,\mathbf{C} \tag{31}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L} \\ 0 & \frac{EI_e}{L^3}p^2 & 0 & -\frac{EI_e}{L^3}q^2 & 0 & 0 \\ 0 & 0 & -\frac{EI_e}{L^2}\tau^2 & 0 & 0 & \frac{EI_e}{L^2}\mu/r^2 \\ 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} \\ 0 & -\frac{EI_e}{L^3}p^2 & -\frac{EI_e}{L^3}q^2\sin\lambda & \frac{EI_e}{L^3}q^2\cos\lambda & 0 & 0 \\ 0 & 0 & \frac{EI_e}{L^2}\tau^2\cos\lambda & \frac{EI_e}{L^2}\tau^2\sin\lambda & 0 & -\frac{EI_e}{L^2}\mu/r^2 \end{bmatrix} \tag{32}$$

By eliminating the constant vector, **A** from Equations (29) and (31), **P** and **δ** can now be related to give the stiffness matrix of the axial-bending coupled column as

$$\mathbf{F} = \mathbf{K}\,\boldsymbol{\delta} \tag{33}$$

where

$$\mathbf{K} = \mathbf{B}\,\mathbf{A}^{-1} \tag{34}$$

It should be noted that the stiffness matrix **K** of Equation (34) will be always symmetric. Now it can be used in conjunction with the Wittrick-Williams algorithm [25] to compute the critical buckling load of axial-flexural coupled columns with the effects of shear deformation.

3 Results and Discussion

The exact stiffness matrix developed above is now applied to an illustrative example which is that of an open section thin-walled square box beam whose cross-section is shown in Figure 4. This example is selected to demonstrate the importance of the proposed theory for a column heavily coupled in axial and flexural deformations. The same problem was previously investigated by Banerjee [26] in the context of free vibration analysis when considering bending-torsional coupling as opposed to axial-bending coupling. Thus, the cross-sectional dimensions of Figure 4 were taken from [26] which are: b = 0.073 m (centre line dimensions) and t = 0.003m. The Young's modulus (E), shear modulus (G) and the shear correction factor (k) were set to 74.55 GPa, 28.06 GPa and 0.66667, respectively. The sectional properties were worked out as, EA=6.5306×10^7 N, EI_e = 4.9846×10^5 Nm^2, kAG= 16.387×10^6 N and z_a = 0.082125m. The length L of the column was taken to be 1 m. The critical buckling loads (P_{cr}) of the column with pinned-simply supported (P-S), clamped-free (C-F) and clamped-clamped (C-C) boundary conditions were computed with and without the effects of shear deformation, and with and without the effects of the axial-flexural coupling parameter μ^2 (see Equation (17)). The results are shown in Table 1. Clearly, both the effects of shear deformation and axial-flexural coupling parameter diminish the critical buckling load, but the axial-flexural coupling effect appears to have a more pronounced effect on the critical buckling load. This will be particularly true for cross-section with a wider separation between the elastic axis and centroidal axis. The results of Table 1 were further checked using the computer program BUNVIS-RG [27] which can analyse buckling behaviour of columns in an exact sense with the inclusion of shear deformation if the data file is appropriately adapted to simulate the axial-flexural coupling effect by eccentrically connecting a column member at the two connecting nodes. The results of Table 1 agreed completely with BUNVIS-RG [27] results confirming the predictable accuracy and correctness of the theory.

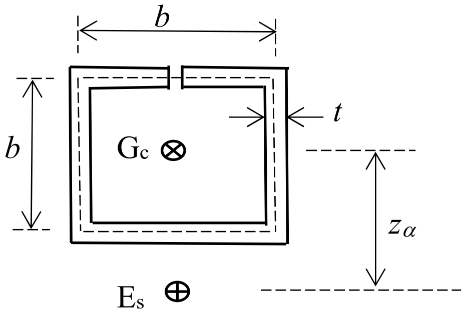

Fig. 4. A thin-walled open square box section

Boundary Condition	P_{CR} (kN)		
	$\mu^2\neq0$, $s^2\neq0$	$\mu^2\neq0$, $s^2=0$	$\mu^2=0$, $s^2\neq0$
C-F	142.38	143.36	3984.0
P-S	558.14	573.43	1161.9
C-C	2067.2	2293.7	10143.8

Table 1. Critical buckling load of a thin-walled open square cross-section column for various boundary conditions.

4 Conclusions

An exact stiffness matrix for an axial-flexural coupled column with the inclusion of shear deformation is developed. The ensuing stiffness matrix is applied with particular reference to the Wittrick-Williams algorithm to investigate the buckling behaviour of a column with substantial coupling between axial and flexural deformation. The investigation has shown that both the shear deformation and axial-flexural coupling effects diminish the critical buckling load, but the latter has a more pronounced effect. The investigation is particularly significant for columns with smaller slenderness ratios and cross-sections having wide separation of centroid and shear centre.

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