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Free vibration of cracked beams using the dynamic stiffness method and Timoshenko-Ehrenfest beam theory

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Abstract

The dynamic stiffness matrix of a cracked Timoshenko-Ehrenfest beam is developed to investigate its free vibration characteristics. The cracked beam is modelled by connecting two intact Timoshenko-Ehrenfest beam elements and an infinitesimal small length cracked element. For the cracked element, the flexibility matrix and subsequent stiffness matrix are established by applying fracture mechanics. The governing differential equations of motion and natural boundary conditions are obtained by applying Hamilton's principle. For harmonic oscillation the equations are solved for displacements and bending rotation. The shear force and bending moment are obtained from the natural boundary conditions. The dynamic stiffness matrix of the intact beam is then derived by relating the amplitudes of loads to those of the displacements. Next, the compliance properties of the crack element are derived using facture mechanics theory. The dynamic stiffness matrices of the three components, namely the two intact elements and one crack element, are assembled to form the overall dynamic stiffness matrix for the cracked beam. The formulation leads to a non-linear eigenvalue problem. The natural frequencies and mode shapes are extracted by applying the Wittrick-Williams algorithm. Results for the cantilever boundary conditions of the cracked beam are presented for illustrative purposes, and the effects of crack location and crack depth on the natural frequencies and mode shapes are examined. Some results are compared with published literature to confirm the validity and accuracy of the proposed method. The theory developed can be extended to include frameworks and other structures.

Keywords: cracked beam, dynamic stiffness method, Wittrick-Williams algorithm, Timoshenko-Ehrenfest beam

1 Introduction

The free vibration analysis of cracked beams has been predominantly investigated using the finite element method as evident from the literature, see, for example [1-2]. However, it is well recognised that the dynamic stiffness method (DSM) [3] provides exact results and therefore, it has much better model accuracy than the finite element and other approximate methods. The main reason for the striking difference is that the shape function used in the DSM is exact unlike the finite element and other approximate methods in which it is generally assumed as a polynomial or interpolation function.

In the current investigation, the free vibration characteristics of cracked beams is analysed by using the DSM and the Timoshenko-Ehrenfest beam theory which accounts for the effects of shear deformation and rotary inertia providing particularly useful and accurate results when the slenderness ratio of the beam is small. First the governing differential equations of motion are developed for the two intact uniform beams which are modelled by using the Timoshenko-Ehrenfest beam theory. The governing differential equations of motion and natural boundary conditions are obtained by applying Hamilton's principle. The next step is to solve the equations for axial and flexural displacements as well as for bending rotation when the oscillatory motion is harmonic. The expressions for shear force and bending moment obtained from the natural boundary conditions as resulted from the Hamiltonian formulation are utilised. The procedure for generating the governing differential equations of motion and natural boundary conditions of the beam achieved from the application of symbolic computation [4]. The dynamic stiffness matrix of the intact beam is derived by relating the amplitudes of loads to those of the responses at the two ends of the beam. This is followed by deriving the compliance properties of the cracked element using facture mechanics theory. Then the overall dynamic stiffness matrix is assembled by connecting the two intact beam elements with the cracked element. Once the dynamic stiffness matrix of the cracked beam is derived, the Wittrick-Williams algorithm [5] is applied as solution technique to obtain its natural frequencies and mode shapes. The results are analysed for various boundary conditions, crack length and crack location. For illustrative purpose, the results for the case of cantilever boundary conditions are presented and discussed in the paper. Finally, the principal findings are summarised.

2 Methods

Figure 1 shows a Timoshenko-Ehrenfest beam of width B, thickness h, and length L. The axial, bending and shear rigidities of the beam are EA, EI, and kAG with the second moment of area I and the Young's modulus E. The mass per unit length, density of material, cross-section area of the beam are m, ρ and A, respectively. The beam is modelled by two intact elements (I and II) of lengths L_1 and L_2 connected by a cracked element (III) of crack depth a at the distance L_1 from the origin. The beam is deflected in the XY plane undergoing axial displacement, bending displacement and bending rotation.

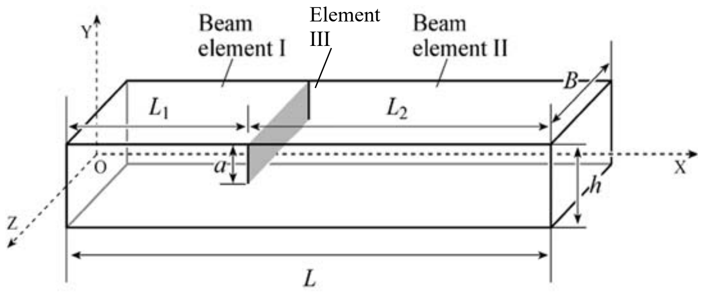

Figure 1: Coordinate system and notation for a cracked beam

i) Intact Beam Modelling

The kinetic and potential energies of the two intact beam elements are obtained using the Timoshenko-Ehrenfest beam theory [6]. Applying Hamilton's principle, the governing differential equations of motions in free vibration are obtained together with the expressions for axial force, shear force and bending moment [4]. Introducing a non-dimensional length and assuming harmonic oscillation, the governing differential equations of motion are derived and solved in terms of two sets of constants which are related to each other. Figures 2 and 3 show the sign convention for axial force, shear force and bending moment, the boundary conditions for displacements and forces respectively.

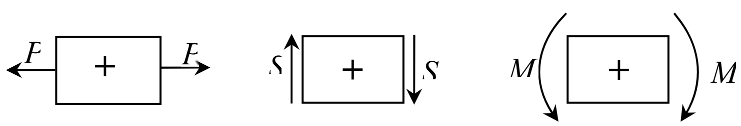

Figure 2: Sign convention for positive axial force P, shear force S and bending moment M.

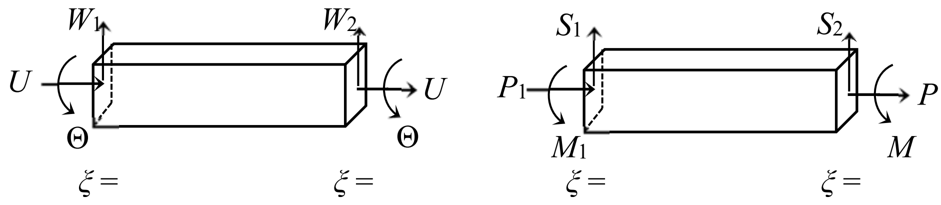

Figure 3: Boundary conditions for displacements and forces.

The dynamic stiffness matrix **K** (6×6) is obtained as follows

$$\mathbf{K}^{\mathrm{I}} = \begin{bmatrix} \mathbf{K}^{\mathrm{I}}_{11} & \mathbf{K}^{\mathrm{I}}_{12} \\ \mathbf{K}^{\mathrm{I}}_{21} & \mathbf{K}^{\mathrm{I}}_{22} \end{bmatrix}, \quad \mathbf{K}^{\mathrm{II}} = \begin{bmatrix} \mathbf{K}^{\mathrm{II}}_{11} & \mathbf{K}^{\mathrm{II}}_{12} \\ \mathbf{K}^{\mathrm{II}}_{21} & \mathbf{K}^{\mathrm{II}}_{22} \end{bmatrix} \tag{1}$$

ii) Cracked Element Modelling

The best fitted formulas for explicit flexibility matrix **C** (3×3) of a cracked element in terms of the cross-sectional dimensions and the crack depth through the thickness are taken from [2]. The dynamic stiffness matrix $\mathbf{K}^{\mathrm{III}}$ (6×6) representing the force displacement relationship at the both ends is constructed:

$$\mathbf{K}^{\mathrm{III}} = \begin{bmatrix} \mathbf{C}^{-1} & -\mathbf{C}^{-1} \\ -\mathbf{C}^{-1} & \mathbf{C}^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{K}^{\mathrm{III}}_{11} & \mathbf{K}^{\mathrm{III}}_{12} \\ \mathbf{K}^{\mathrm{III}}_{21} & \mathbf{K}^{\mathrm{III}}_{22} \end{bmatrix} \tag{2}$$

iii) The Whole Cracked Beam System

The overall dynamic stiffness matrix $\mathbf{K}(\omega)$ is assembled using $\mathbf{K}^{\mathrm{I}}$, $\mathbf{K}^{\mathrm{I}}$ and $\mathbf{K}^{\mathrm{III}}$ as:

$$\mathbf{K}(\omega) = \begin{bmatrix} \mathbf{K}^{\mathrm{I}}_{11} & \mathbf{K}^{\mathrm{I}}_{12} & 0 & 0 \\ \mathbf{K}^{\mathrm{I}}_{21} & \mathbf{K}^{\mathrm{I}}_{22} + \mathbf{K}^{\mathrm{III}}_{11} & \mathbf{K}^{\mathrm{III}}_{12} & 0 \\ 0 & \mathbf{K}^{\mathrm{III}}_{21} & \mathbf{K}^{\mathrm{III}}_{22} + \mathbf{K}^{\mathrm{II}}_{11} & \mathbf{K}^{\mathrm{II}}_{12} \\ 0 & 0 & \mathbf{K}^{\mathrm{II}}_{21} & \mathbf{K}^{\mathrm{II}}_{22} \end{bmatrix} \tag{3}$$

The Wittrick-Williams algorithm is used as a solution technique in solving the transcendental eigenvalue problem as in the present case. Boundary conditions are applied by deleting the rows and columns of $\mathbf{K}(\omega)$ corresponding to zero displacements and rotations when computing the natural frequencies and mode shapes of individual cases such as cantilever, simply-supported and clamped-clamped cracked beams. A non-uniform cracked beam can be analysed for its free vibration characteristics by assembling many uniform cracked beams.

3 Results

Numerical results of a cracked Timoshenko beam are obtained for different boundary conditions. The data are taken from [1]: E=216GPa, ρ=7850kgm^{-3}, ν=0.28, L=0.2m, B=0.025m, h=0.0078m. For illustrative purposes, the results are presented for the cantilever boundary conditions. The natural frequencies and mode shapes are obtained with respect to different crack location, crack depth. A few selected results are presented here.

Excellent agreement with published results as shown in Table 1 can be observed. The fundamental natural frequency ratio which is the ratio of natural frequencies of the cracked and intact beam, are plotted in Figure 4. The natural frequencies of the cracked beam are lower than the corresponding intact ones, as expected. The difference naturally increases with the crack depth. It is clear when a crack nearer to the built-in end has a much greater effect than the one located nearer to the free end because the maximum bending moment occurs at the built-in end and the bending moment gradually reduces towards the tip and finally becomes exactly zero at the tip. The natural frequencies are almost unchanged when the crack is far away from the fixed end. Figure 5 shows the fundamental natural frequency ratio against the crack depth ratio for two representative crack location ratios.

4 Conclusions and Contributions

The free vibration characteristics of a cracked Timoshenko-Ehrenfest beam has been investigated using the dynamic stiffness method. The cracked beam is idealised by two intact Timoshenko-Ehrenfest beam elements and a cracked element. The dynamic stiffness matrix is formulated through an assembly procedure combining cracked and intact elements. The formulation resulted in a nonlinear eigenvalue problem which was solved by applying the Wittrick-Williams algorithm. Numerical results are presented for cantilever boundary conditions to serve as an illustrative example. The effects of crack location and crack depth on the free vibration behaviour are discussed and representative mode shapes are presented. The accuracy of results using the dynamic stiffness method is an important attribute to this research. The results provided benchmark solutions so that the finite element and other approximate methods can be calibrated. The research carried out in this paper, is expected to pave the way for further research on complex structural systems containing crack elements.

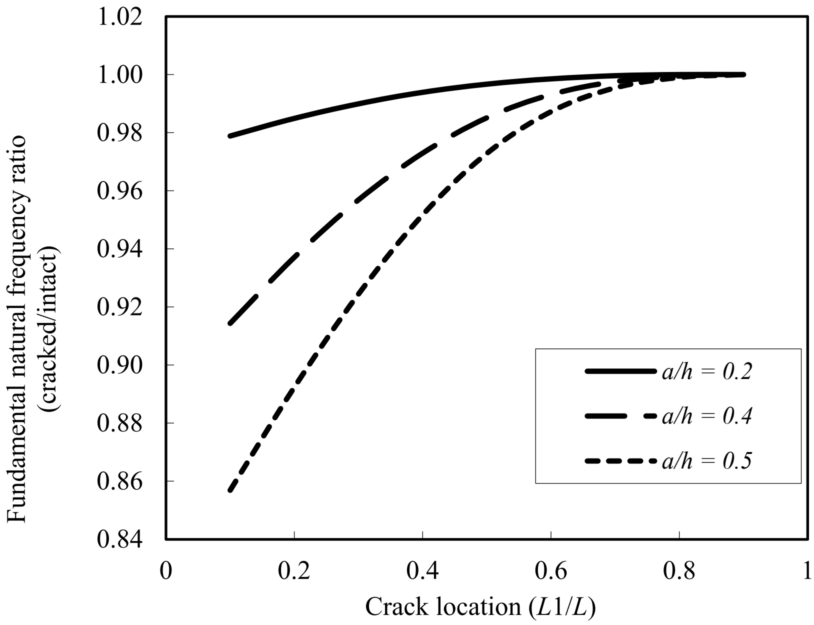

Figure 4: Fundamental natural frequency ratio against the crack locations with *a/h*.

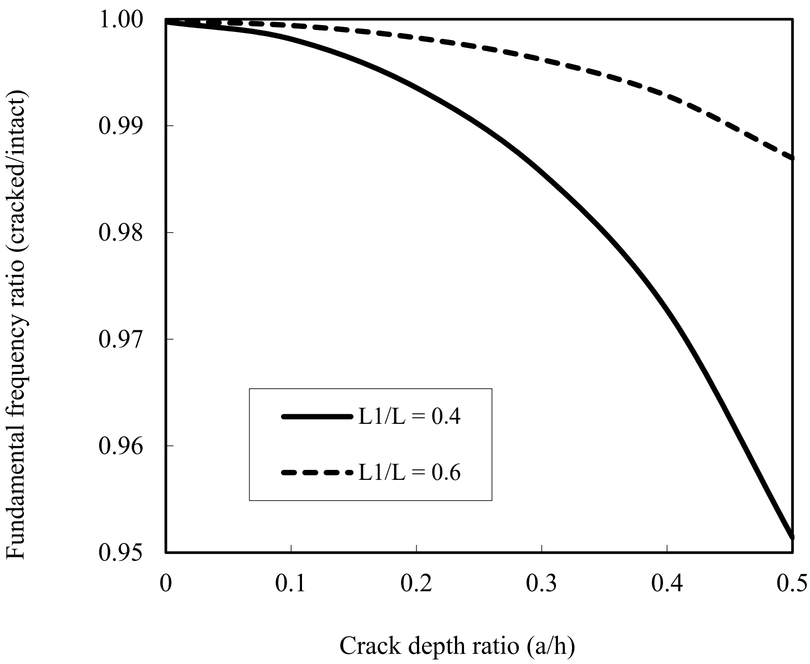

Figure 5: Fundamental natural frequency ratios against a/h for L_1/L sets at 0.4 and 0.6.

L_1/L	ω_i (rad/s)	*a / h*					
		0.2		0.4		0.6	
		Current	Ref [1]	Current	Ref [1]	Current	Ref [1]
0.2	1	1019.916	1020.137	970.815	966.9525	852.480	842.2205
	2	6420.571	6457.396	6424.429	6454.483	6417.650	6448.175
	3	17651.16	17872.91	17433.77	17596.57	16876.49	16944.56
	4	33674.48	34553.13	32647.85	33100.42	30488.91	29796.26
0.4	1	1029.176	1030.095	1007.670	1006.856	947.811	942.7322
	2	6358.443	6389.394	6167.999	6174.539	5702.459	5689.841
	3	17608.81	17844.86	17326.41	17499.83	16611.97	16792.25
	4	33986.26	34866.97	33767.44	34420.09	33017.86	32971.51
0.6	1	1034.025	1035.284	1028.455	1029.262	1011.412	1010.864
	2	6339.555	6365.914	6064.305	6071.655	5400.865	5371.803
	3	17600.06	17807.94	17197.25	17359.27	16401.77	16478.82
	4	34026.17	34895.50	33713.21	34572.37	33073.52	33710.43
0.8	1	1035.504	1036.884	1035.067	1036.414	1033.700	1034.943
	2	6414.198	6440.057	6353.616	6375.921	6163.768	6174.710
	3	17595.15	17758.61	16951.02	17077.99	15255.65	15286.83
	4	33674.59	34393.87	32081.06	32639.52	29281.01	29529.79
Intact Beam	1	1035.812	1037.0189				
	2	6428.052	6458.3438				
	3	17778.74	17960.564				
	4	34279.67	34995.429				

Table 1: The first four natural frequencies of a cracked beam with L_1/L and *a/h* for cantilever boundary condition.

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