Proceedings of the Sixteenth International Conference on
Computational Structures Technology
Edited by: P. Iványi, J. Kruis and B.H.V. Topping
Civil-Comp Conferences, Volume 14, Paper 1.1
Civil-Comp Press, Edinburgh, United Kingdom, 2026
ISSN: 2753-3239, doi: 10.4203/ccc.14.1.1

Challenges in Topology Optimization with Applications

J. Lógó, M. Tantawi and B. Tóth

Faculty of Civil Engineering,
Budapest University of Technology and Economics
Hungary

Abstract

Optimization is one of the most intensively investigated research area in engineering and needs very complex knowledge in the mathematics, physics and informatics. Topology optimization has been the most popular subfield of it recently. The early period achievement was connected to deterministic design, however recently the probabilistic problems become also widely investigated. It has more than 150 years of history. The starting date can be connected to the Maxwell's paper published in 1870. This presentation is an overview of subjectively selected state-of-art achievements during the full history what is more than 150 years. The selection of the achievements of this field is a rather difficult task because in the early period of the history of topology optimization lot of meetings were classified and the results were not available for public. The optimization community has almost no knowledge about the publications in topology optimization in the 50s of the past century. This presentation also gives an overview of the recent achievements with industrial applications as, well. These applications are based on our recent researches.

Keywords: topology optimization, optimal design, truss-like structure, deterministic design, probabilistic design, reliability based optimization.

1 Introduction

All fields of the life the most economical solution is primary important. In engineering it could mean one has to find a design which leads for the strongest structure with lightest weight. To interpret the above two sentences in scientific way one has to harmonize his/her knowledge in mathematics, physics/mechanics and computer science. In mathematics the operational research gives appropriate tools to find the optimal solution. Combining the achievements of the mathematical programming with physical/mechanical theories one can be in the field of structural optimization. Depending on stochastic nature of the required design data, the structural optimization can be used for deterministic

Structural optimization has a long history, starting with the work of Galileo, however its subfield, the topology optimization is relatively young with its 156 years background. It took several decades to overcome the initial difficulties. Farkas' theorem as the key principle in linear programming provided and appropriate tool to find the best solution, but the mechanical applications were very limited. In the end of 1930s Waiuntyski's theorems [33] was the only sign that the structural optimization is still alive. Starting from the 1950s, the mathematical foundations (e.g. Kuhn-Tucker theorem) and the mechanical foundations (e.g. the extreme value theorems of plasticity) were published, but the basic computational tools were lacking. Outstanding theoretical results were achieved, but the "advancement" of numerical methods did not allow for the widespread application of optimal design. The 1970s brought about a leap in this field of science as well. The wider application of the extreme value theorems of plasticity resulted in an explosive development of this field.

Topology optimization as a subfield of the structural optimization is the most popular research area recently. One can find applications in all fields of life, including natural sciences and medical problems. The starting date could be derived from the Maxwell's paper was published in 1870. His results were generalized later by Michell. It is interesting that Michell's optimization paper was considered as the first publication in topology design from 1904.

This conference paper is an overview of subjectively selected state-of-art achievements during the full history and based on the first author earlier papers [12-18]. The selection of the achievements of this field is a rather difficult task because in the early period of the history of topology optimization lot of meetings were classified and the results were not available for public. The optimization community has almost no knowledge about the publications in topology optimization in the 50s of the past century. After the 70s of the past century this situation has changed and one has more possibilities to find more publications due to the changes and thanks to the digitalization.

According to the literature search of the theorem of the optimal design, the topology design remained unnoticed for some fifty years until the publication of the papers of Foulkes in 1954, Cox 1958 and Hemp 1958. Among others the papers written by Sved [30] and Barta [2] are the most significant ones. The two works have lot of similarity in contents and in conclusions. In Barta's paper, the minimum volume design of plane and space structures (trusses) were discussed. He proved that by removing given number of properly chosen redundant bars from a given network, it is possible to obtain such a statically determinate structure, which yields a structure with the least weight. It has to be noted that this conclusion was stated for single and deterministic loads, as well. Barta also concluded that the proof did not guarantee that only statically determinate structure could be the least weight solution. Sved's paper can be considered as the origin of the stress limited minimum volume design.

The true blow-up in layout theory was in the 60's and the 70's. In the 60's, the significant publications helped to derive optimality conditions for minimum volume designs. Shield in 1973 presented optimum design methods for multiple loading. He used variational principles to prove the optimality. In 1960 Schmit applied full stress design (FSD) to statically indeterminate structures and found that FSD provides exact optimum in a single sizing operation for statically determinate structures where the internal forces remain constant during resizing, but for indeterminate structures, the number of resizing iterations can vary from a few to many as a function of the sensitivity of internal forces to changes in member sizes. The reason why these different results were obtained is known now: it mainly depends on the redundancy of structures or in other words, it depends on whether the internal forces remain almost constant during resizing, as it happens in most well designed practical structures. Berke and Khot [4] (in 1974) concluded that minimum weight design should be at the point including "fully stressed elements, lower bound elements, and neither of them".

In the period of 1960-80 can be named as the origins of the exact structural topologies. Nagtegaal and Prager in 1973 investigated the optimal truss layout theory in case of alternative loads. Prager in 1971 derived an optimality condition for beams and frames subjected to alternating loading by the use of Foulkes mechanism. Prager and Rozvany in 1977 extended the existing optimal layout theory originally used for low volume fraction, to grid-like structures (trusses, grillages, shell-grids etc.). The method was validated for the case of not restricting to low volume fraction structures. Rozvany et al. [26] provided solution on exact optimal topologies of perforated plates. The field of truss topology design was extended by the investigation of Achtziger at late 80's and 90's of the last century. In layout and size optimization the three bar truss example was investigated by several authors due to the complexity of the problem.

The origins of the numerical solution technique of the constrained optimality criteria methods (COC) were published by Berke and Khot in 1974. This provided the mathematical background of an effective solution technique in topology optimization. The first numerical procedure for finite element (FE) based topology optimization was elaborated by Rossow and Taylor [24] in 1973, but the real expansion started at the end of the 80's by Bendsoe and Kikuchi [3] and Rozvany [27].

Almost two decades later the probabilistic topology optimization was a new direction in this field. In the first two decades of this century at first Kharmanda et al. [11] published a reliability based topology paper. Later, Lógó et al. [13-16], Guest et al. [7] Dunning et al. [8] and Csébfalvi [7] introduced some efficient and accurate approaches to probabilistic and/or robust structural topology optimization. Generally, the objective is to minimize the expected compliance or volume with uncertainty in loading magnitude and applied direction, where uncertainties are assumed normally distributed values and statistically independent. This approach is analogous to a multiple load case problem where load cases and weights are derived analytically to accurately and efficiently compute the expected compliance and sensitivities. In the following some selected achievements are presented in more details.

2 Challenges in topology optimization

Here some basic achievements are presented which have significant influence for the entire topology optimization. Among them has to be mentioned the fundamental mathematical theorem of Gy. Farkas in connection with the linear programming. Also important result is the plastic hinge theorem by Kazinczy [10]which has fundamental role in limit design. Kazinczy was also among the first researchers who investigated the problem of the statically indeterminate trusses in case of multiple load conditions. Using the Cremona-type solution procedure, he investigated the case of the pre-stressing technique to reach the uniform collapse of the member forces in the case of statically indeterminate structures. With this technique he used the shakedown theory without naming it, much earlier than Melan published it in 1936. This Cremona-type solution technique has recently been used again as a numerical procedure in topology design. Kazinczy also discussed the questions of safety and reliability designs much earlier than anybody else in the world.

Maxwell's theorem and Michell's formulation on minimum volume design

The Maxwell's problem [19] can be described as follows: consider a truss which maintains equilibrium with a set of forces $\overline{F_\iota}$ acting at the points forces $\overline{r_\iota}$, (i=1, 2,…, n). Denote by T_t the load carried in a typical tension member with length L_t and the section area A_t while in the case of typical compression members these variables are T_c , L_c, and A_c. The permissible stresses were denoted by f_t and f_c , respectively. By the use of the principle of virtual work, Maxwell derived the optimality condition of the lightest structure, which has the volume given by:

$$V = V_c\left(1+\frac{f_c}{f_t}\right)+\frac{1}{f_t}\sum_i \overline{F_i}\overline{r_i} = V_t\left(1+\frac{f_t}{f_c}\right)-\frac{1}{f_c}\sum_i \overline{F_i}\overline{r_i} \tag{1}$$

here V_t is the volume of all the tension members and V_c is the volume of all the compression members.

Michell generalized the above theorem to get his formulation [20]. The Michell problem can be described as follows: in addition to the Maxwell's problem above, let us consider a series of external forces $\overline{F_\iota}$ acting at the points forces $\overline{r_\iota}$, (i=1, 2,…, n). Let D be a domain of space containing the points $\overline{r_\iota}$, (it should be noted that D can be the whole of feasible space). Consider then all possible frameworks (trusses) S, contained in D, which equilibrate the forces $\overline{F_\iota}$ and which satisfy the limiting conditions on stresses. Let us assume that there is a framework S* which satisfies the following condition of Michell: "There exists a virtual deformation of the domain D such that the strain along all members of S* is equal to ±e, where e is a small positive number, and where the sign agrees with the sign of the end load carried by the particular member, and further that no linear element of D has strain numerically greater than e.” Michell’s theorem states that the volume V* of S* is less than or equal to the volume V of any of the frameworks S.

$$V = \frac{(f_t + f_c)}{2 f_t f_c}\left(\sum_t L_t T_t + \sum_c L_c T_c\right) - \frac{(f_t - f_c)}{2 f_t f_c}\sum_i \bar{F}_i \bar{r}_i \quad . \tag{2}$$

The actual value of V^* follows from the principle of virtual work. If the virtual displacements corresponding to Michell’s statement above are $e\bar{v}_i$ at $\bar{r}_i$ this volume V^* is:

$$V^* = \frac{(f_t + f_c)}{2 f_t f_c}\sum_i \bar{F}_i \bar{v}_i - \frac{(f_t - f_c)}{2 f_t f_c}\sum_i \bar{F}_i \bar{r}_i \quad . \tag{3}$$

It is noted that the original formulas of Michell are not valid for different allowable stresses in tension and compression.

Cox’s optimal solutions

Applications of the theorems of Maxwell and Michell to simple design problems have been made by Cox [6]. He has considered first of all the problem of three coplanar forces. In the case where their point of intersection lies within the triangle formed by their points of application, the optimum framework can consist of tension or compression members only. Some of his layouts are given in Figure 1. and we have to note that all these structures have equal weights. One can recognise the non-uniqueness of the optimal layout and we can have an infinite number of solutions ranging from mechanisms through simple stiff structures to structures of any degree of redundancy.

Cox extended the theory presented above to build a structure for the transmission of a bending moment. He showed that if the proportion of length over height of the structure is greater than 4 this structure is considerably lighter than a “simple tie and strut” and that for larger values of length/height multiple constructions, on the lines of Figure 1. can be even lighter. He produced a competitive 14-bar framework and a variation of Figure 1., in which the circles are replaced by spirals, which for length/height > 4 is lighter than any other construction considered. These structures for the transmission of bending moments are not Michell’s optimum structures, since they fail to satisfy the orthogonality conditions for members with opposite signed loads.

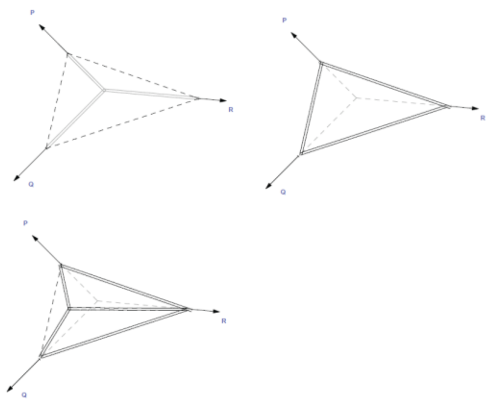

Figure 1: Simple tension structures by Cox

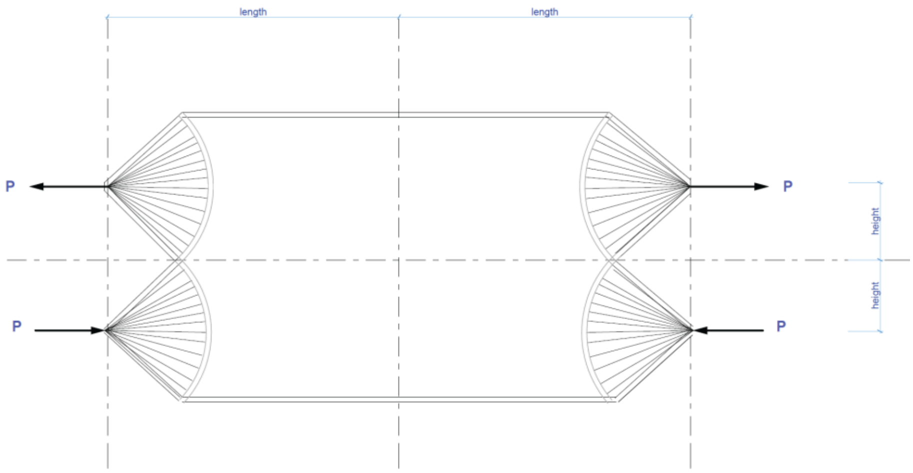

Figure 2.: Cox's optimal beam for bending

Design dependence of the optimal solutions in the case of alternating loads

In optimization the investigation of the convergence of the applied procedure and the uniqueness of the solution are of primary importance, however the non-uniqueness gives opportunities to the designers. This question is more difficult whenever there are multiple loadings and/or the loading uncertainty is considered. In this latter case the load can be considered as a quantity given in an interval with a certain possibility of location or/and direction, and/or magnitude. The design loads are usually selected on the base of worst loading cases. To understand the topic here, the main achievements of the paper of Nagtegaal and Prager [21] are discussed briefly here. The starting point is a minimum volume design of a truss. The results of this layout optimization, coming from this uniaxial case, can be generalized and a continuum type topology optimization problem with bi-axial stress state is investigated later. The paper of Nagtegaal and Prager is concerned with the following problem: two alternative loads with the same point of application are to be transmitted to a rigid foundation by a plane truss of minimum weight whose load factors for plastic collapse under one or the other load are not to exceed a given value. A necessary and sufficient condition for global optimality is established and used to determine the optimal layout of the truss. According to their optimality criteria (global optimality condition) in an active truss member (being non zero cross-section) the sum of the normalized strain rates is unit, while in vanishing members, the summation of the strain rates results in a smaller value than unit. In addition to the optimality conditions Nagtegaal and Prager gave a short overview how the optimal layout looks like in a special case of alternative loads (see Figure 3.). They considered a fixed force (say P') and discussed the optimal types of trusses for all possible other forces (P"). In Figure 3. "A" is the common application point of these two forces. Here vector AB' represents the fixed force P'. The lines B'C and B'D form angles of 45° with the horizontal direction, and lines EF and EG are obtained by mirroring the lines B'C and B'D with respect to point "A".

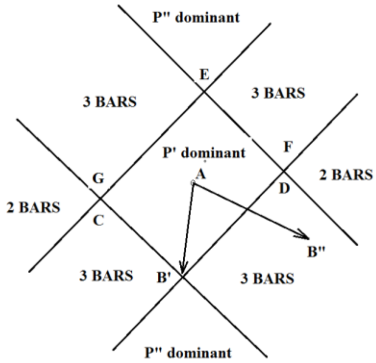

Figure 3: Truss type as function of the alternative loading

These lines divide the plane of the figure into nine regions. If the load P” is represented by the vector AB”, the label of the region that contains B” indicates the type of the optimal truss. Where one of the loads is dominant, that exclusively determines the optimal design, the optimal truss consists of two bars that form angles of 45° with the wall.

The general layout theories can act as reference studies for the case of topology optimization of truss-like structures in the case of multiple and/or stochastic loading. It is to note that in the case of trusses the uniaxial stress state is considered, but the truss-like structures belong to the biaxial stress state problems.

Approximation of a Probabilistic Expression

According to the approximation theory of Kataoka [21] a stochastic expression can be upperbounded by a convex deterministic one. The outline of this method can be explained as follow: if $\xi_1, \xi_2, ..., \xi_n$ have a joint normal distribution, then the set of $\mathbf{x} \in \Re^n$ vectors satisfying

$$\mathrm{P}(x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n \leq 0\ ) \geq q \tag{4.a}$$

is the same as those satisfying

$$\sum_{i=1}^{n} x_i\mu_i + \Phi^{-1}\left(q\right)\sqrt{\mathbf{x}^T\mathbf{K}_{\mathrm{ov}}\mathbf{x}} \leq 0 \tag{4.b}$$

where $\mu_i = E\left(\xi_i\right)$, $(i = 1,2,...,n)$ is the mean value of the randomly given element ξ_i, $\mathbf{K}_{\mathrm{ov}}$ is the covariance matrix of the random vector $\boldsymbol{\xi}^T = \left(\xi_1, \xi_2, ..., \xi_n\right)$, q is a fixed probability and $0\langle q \langle 1$, $\Phi^{-1}\left(q\right)$ is the inverse cumulative distribution function (so called probit function) of the normal distribution. Expression (4.b) is convex, the proof can be found in Prekopa [21]. According the original approximation theory of Kataoka the probit function is substituted by an appropriate constant and the Gaussian distribution is not a requirement.

The theorem above can be used in probabilistic topology optimization where the uncertain compliance value can be approximated [13-15].

3 Selected applications

Optimal design of grid systems for tall buildings using lattice structures

Diagrids and hexagrids are truss structures that employ inclined members instead of vertical columns to carry both vertical and lateral loading [5]. Perimetral grids are an effective way to deal with horizontal forces in tall buildings. Diagrids consist of members that diagonally intersect to generate triangular shapes; hexagrids were inspired by honeycombs, where hexagonal patterns constitute the resisting bulk. Mechanical properties of such kind of grids were investigated and critically review both in the scientific and technical literature. Their optimization is generally performed by using size optimization or homogenization in conjunction with a cantilever beam model of the high-raise building.

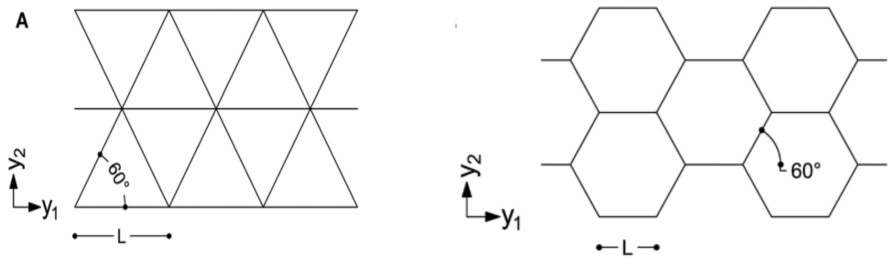

Figure.4.a.(A) an isotropic triangular lattice of a diagrid; (B) an isotropic hexagonal lattice of a hexagrid

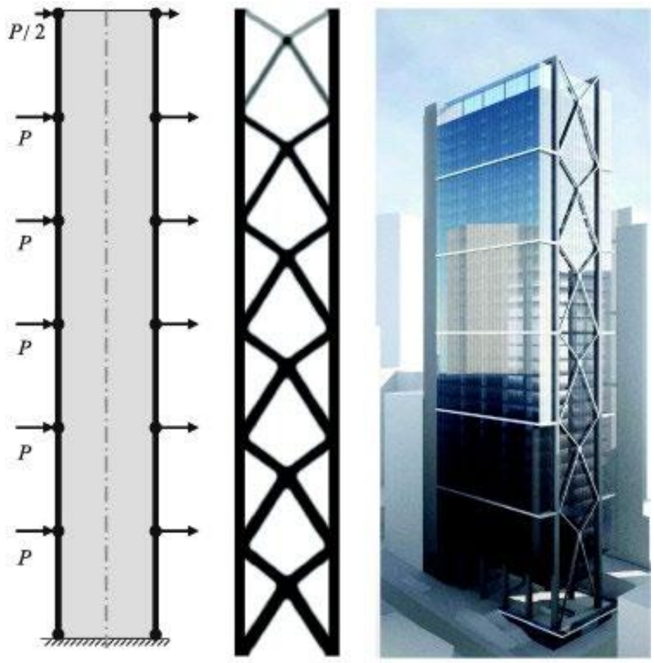

Figure 4.b. Illustration for the concept design of a 288 m tall high-rise in Australia, showing the engineering and architecture expressed together at Skidmore, Owings & Merrill LLP (Beghini, Katz, Baker and Paulino, 2014)

This section addresses the optimal design of grid systems in high-rise buildings by means of topology optimization and multiscale analysis of periodic solids. Instead of adopting a multiscale beam model, a 3D box-shaped finite element mesh is used as a discrete design domain to seek for optimal grids whose panels can be regarded as lattice structures. Among the others, computes analytical expressions for the constitutive tensors that describe the elastic behavior of triangular and hexagonal lattices at the macroscopic level. Such expressions depend on the prescribed reference dimension, cross-section, and material used at the mesoscopic level (the cell). A multi-material topology optimization problem is used that employs continuous variables to distribute a discrete set of lattices. Each phase stands for a candidate lattice that has given features in terms of constituent elements and geometrical properties. The stated problem searches for the distribution of cross-sections and shapes that minimizes the weight of the grid under enforcements on the stiffness of the building under lateral loads (displacement constraints at the top of the building). In order to improve manufacturability of the achieved layouts, patches are introduced to force the distribution of the same lattice at a given height or within a minimum contiguous area. This also helps in reducing the number of optimization variables used in the optimization.

Reliability based optimal design based on optimality criteria algorithm

This section [32] formulates probabilistic topology optimization problem as a size optimization under stress constraints and displacement constraints . The derivation of the method starts with general formulation of structural optimization:

$$\begin{aligned} &\text{Find } \boldsymbol{x} = [x_1, x_2, \dots, x_M]^T \text{ which minimizes } f(\boldsymbol{x}) \\ &\text{subject to the constraints } \quad g_i(\boldsymbol{x}) = 0,\ i = 1,2,\dots,N \\ &\qquad\qquad\qquad\qquad\qquad\quad h_j(\boldsymbol{x}) \leq 0,\ j = 1,2,\dots,P \end{aligned} \tag{1}$$

where

$\boldsymbol{x}$ is an M- dimensional vector called the ***design vector***

$f(\boldsymbol{x})$ is termed the ***objective (cost) function***

$g_i(\boldsymbol{x})$ and $h_j(\boldsymbol{x})$ are known as ***equality*** and ***inequality constraints***, respectively.

In the proposed approach the objective function $f(\boldsymbol{x})$ represents overall volume of the structure, equality constraints $g_i(\boldsymbol{x})$ are equilibrium equations and the inequality constraints $h_j(\boldsymbol{x})$ depend on the specific problem. Graphical representation of the initial and optimal topology is shown in Figure 5.

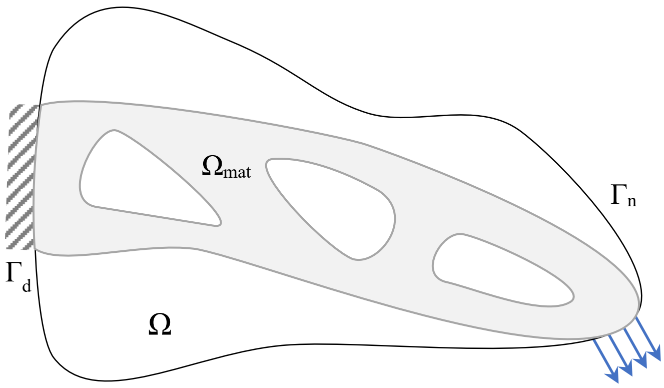

Figure 5: Design domain Ω vs optimal topology Ω_{mat}

Three particularly useful and frequently used type of inequality equations are:

1) $|\boldsymbol{\sigma}(\mathbf{x})| - \sigma_0 \le 0$, in the case of stress constrained optimization
   where $\boldsymbol{\sigma}(\mathbf{x})$ denotes stress concentration at the optimal solution and σ_0 is allowable stress level

2) $\Pr(\boldsymbol{\sigma}(\mathbf{x})) - R_0 \le 0$, in the case of reliability based optimization
   where $\Pr(\sigma(\mathrm{x}))$ represents probability of occurrence of given stress level at the optimal topology and R_0 is a threshold value

3) $N_\mathrm{R}(\mathbf{x}) - N_\mathrm{t} \le 0$, in the case of low-cycle fatigue
   where $N_\mathrm{R}(\mathbf{x})$ is number of cycles at optimal design and N_t represents permissible value

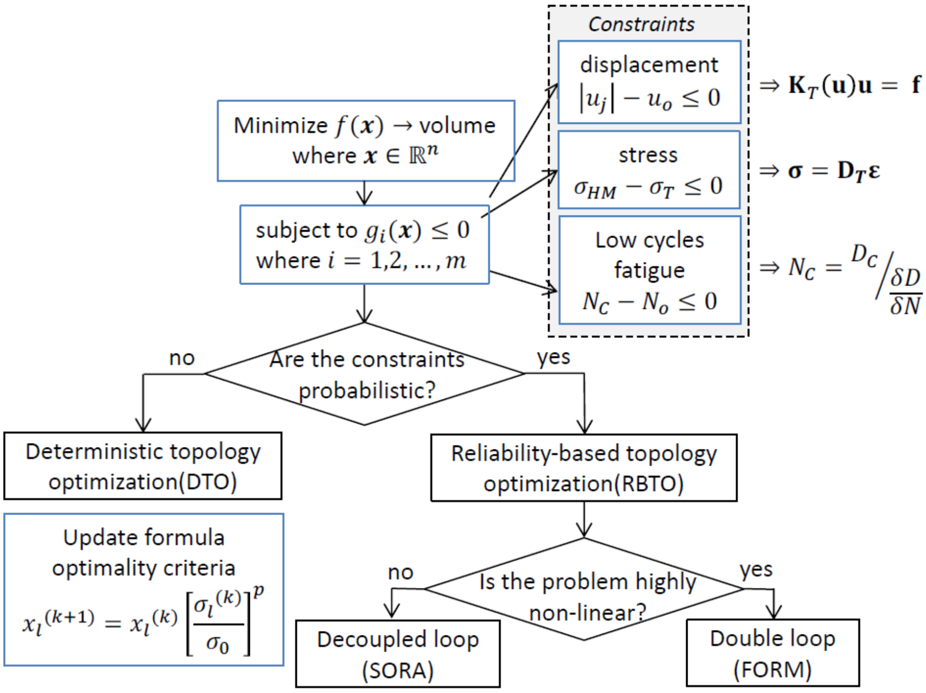

Figure 6: Computational algorithm

Optimal design of spherical like grid shells with uncertain nodal positions

Dome structures as architectural objects were used frequently for covering rather representative buildings during the history. There are several spherical like grid surface divisions but few of them preferable as optimal for the existing structural schemes of shells [31]. Finding the optimal topologies/shapes always provided challenging task for researchers [25]. The classical numerical methods (among them funicular analysis [22], force density method [28]) provide appropriate tools to find most suitable solution for this kind of problems. Recently, the Authors developed an extended version of force density method.

A numerical approach based on extended force density method is elaborated in this contribution to cope with the design of spherical like grid shell structures. The equilibrium of funicular networks can be conveniently handled through the force density method, that means writing the problem in terms of the ratio of force to length in each branch of the spatial structure. As the continuation of Buggi's developments [1], the optimal design of networks with fixed plan geometry is stated both in terms of any independent subset of the force densities and in terms of the height of the restrained nodes. Local enforcements and side constraints are formulated to prescribe lower and upper bounds for the vertical coordinates of the nodes. Due to the nature of the uncertain nodal position the Authors' earlier numerical developments is used to solve the probabilistic constrained optimization problem. Different objective functions are tested, including structural compliance and Maxwell number (which is the sum of the force-times length products for all the branches in the spatial network). A sequential convex programming code is used to get the solution. By the use of the first order optimality conditions the optimality is investigated. The new class of optimal topologies with their numerical confirmation are presented.

4 Conclusions and Contributions

One can see that the achievements of different scientific fields could result new research areas which could lead for better engineering solutions.

Acknowledgements

The second and the third author would like to thank supports from the Ministry of Culture and Innovation of Hungary, through the National Research, Development and Innovation Fund (KDP-2023, C2316739 and EKÖP_KDP-24-1-BME-24 founding schemes).

References

[1] D. Aita, M. Bruggi," A numerical method for the funicular analysis of masonry vaults accounting for stereotomy, finite strength and finite friction", Engineering Structures, Volume 293, 116658, 2023.

[2] J. Barta, "On the Minimum Weight of Certain Redundant Structures", Acta Tech. Acad. Sci. Hung.,V. 18, 67-76, 1957.

[3] M.P. Bendsoe, N. Kikuchi, " Generating Optimal Topologies in Structural Design Using a Homogenization Method", Comp. Meth. Appl. Mech. Eng., 71, 197-224, 1988.

[4] L. Berke, N.S. Khot, "Use of Optimality Criteria Methods for Large Scale System", AGARD Lec.70, 1-29, 1974.

[5] M. Bruggi, H. Ismail, J. Lógó, "Topology Optimization with Graded Infill Accounting for Loading Uncertainty". Composite Structures 311 (2023): 116807.

[6] H. L. Cox, "Structures of Minimum Weight: The Basic Theory of Design Applied to the Beam under Pure Bending", Aeronautical Research Council 19785, 1958.

[7] A. Csébfalvi, "A new theoretical approach for robust truss optimization with uncertain load directions", Mechanics Based Design of Structures and Machines, 42, 4, 442-453, 2014.
[8] P.D. Dunning, H.A. Kim, G. Mullineux, "Introducing Loading Uncertainty in Topology Optimization", AIAA Journal, 49(4), 760-768, 2012.
[9] J.K. Guest, T. Igusa, "Structural optimization under uncertain loads and nodal locations", Comp. Meth. Appl. Mech. Eng., 198, 116-124, 2008. https://doi.org/10.1016/j.cma.2008.04.009
[10] G. Kazinczy, "Experiments with Clamped End Beams" (in Hungarian) Betonszemle (Concrete Review), 2, No.4., 5.,6., 68-71, 83-87, 101-104, 1914.
[11] G. Kharmanda, N. Olhoff, A. Mohamed, M. Lemaire, "Reliability-based topology optimization", Structural and Multidisciplinary Optimization, 26(5), 295-307, 2004.
[12] J. Lógó, " New Type of Optimal Topologies by Iterative Method", Mechanics Based Design of Structures and Machines, 33, 2, 149-172, 2005.
[13] J. Lógó, " New Type of Optimality Criteria Method in Case of Probabilistic Loading Conditions", Mechanics Based Design of Structures and Machines, 35, 2, 147-162, 2007.
[14] J. Lógó, M. Ghaemi, M. Movahedi Rad, "Optimal topologies in case of probabilistic loading: The influence of load correlation", Mechanics Based Design of Structures and Machines, 37 , 3, 327-348 , 2009.
[15] J. Lógó, " SIMP type topology optimization procedure considering uncertain load position", Periodica Polytechnica Civil Engineering, (56)2, 213-219, 2012.
[16] J. Lógó, E. Pintér, "Numerical Methods in Probabilistic Topology Optimisation: A Review", In: B H, V Topping et.al. (eds.) Computational Technology Reviews Stirling, United Kingdom / Scotland: Saxe-Coburg Publications, 79-108, 2012.
[17] J. Lógó, H. Ismail, "Milestones in the 150-year history of topology optimization: a review", Computer Assisted Methods in Engineering and Science, 27(2-3), 97-132, 2020.
[18] J. Lógó, B. Balogh; E. Pintér, "Topology Optimization Considering Multiple Loading", Computers & Structures, 233-244, 2018.
[19] J.C. Maxwell, "On Reciprocal Figures, Frames, and Diagrams of Forces", Transactions of the Royal Society of Edinburgh, 26, (1), 1–40, 1870.
[20] A.G.M. Michell, "The Limits of Economy of Material in Frames Structures", Philosophical Magazine, 8, 589-597, 1904.
[21] C. Nagtegaal, W. Prager, "Optimal Layout of a Truss for Alternative Loads", International Journal of Mechanical Sciences, 15, 7, 583-592, 1973.
[22] D. O'Dwyer, "Funicular analysis of masonry vaults", Computers and Structures, 73(1-5), 187-197, 1999.
[23] A. Prékopa, "Stochastic Programming", Akadémia Kiadó and Kluwer, Budapest, Dordrecht, 1995.
[24] M.P. Rossow, J.E.; Taylor, "A Finite Element Method for the Optimal Design of Variable Thickness Sheets", J. AIAA, 11, 1566-1569, 1973.

[25] G.I.N. Rozvany, W. Prager, “A new class of structural optimization problems: optimal archgrids”, Computer Methods in Applied Mechanics and Engineering, 19, 127-150, 1979.
[26] G.I.N. Rozvany; N. Olhoff, M.P. Bendsoe, T.G. Ong, R. Sandler, W.T. Szeto, “Least-weight design of perforated elastic plates”, DCAMM Report No 306. Int. J. Solids Struct. 23, 521- 536, 537-550, 1985/87.
[27] G.I.N. Rozvany,”Structural Design via Optimality Criteria”, Kluwer Academic Publisher, Dordrect, 1989.
[28] H.J. Schek, “The force density method for form finding and computation of general networks”, Computer Methods in Applied Mechanics and Engineering, 3, 1, 115-134, 1974
[29] R. T. Shield, “Optimum Design of Structures Through Variational Principles”, in Proc. of Optimization and Stability Problems in Continuum Mechanics, ed. by. P.K.C. Wang, Springer-Verlag, Berlin, Heidelberg, New York, 13-37, 1973
[30] G. Sved, “The Minimum Weight of Certain Redundant Structures”, Australian Journal of Applied Sciences, 5, 1-9, 1954.
[31] T. Tarnai (ed.), “Spherical Grid Structure, Geometric Essays on Geodesic Domes”, Hungarian Institute for Building Science, Budapest, 1987.
[32] P. Tauzowski, B. Blachowski, J. Lógó. "Computational framework for a family of methods based on stress-constrained topology optimization." Computers & Structures 303 (2024): 107493.
[33] Z. Wasiutynski, “Strength Design. Part I: Methods of Strength Design. Part II: Design, for Minimum Potential. Part III: On the Design of I-Beams” (in Polish). Akademia Nauk Technicznych, Warsaw, 1939.