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A Finite Element Formulation for Vibration Analysis of Laminated Composite Thin-Walled Beam Type Structures Under Initial Load

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Abstract

This work presents an improved beam formulation for predicting the natural frequencies of composite thin-walled beam-type structures subjected to initial loading. Each wall of a beam element cross-section is idealised as a thin, symmetric and unbalanced angle-ply laminate. The formulation is based on Hooke's law and a geometrically nonlinear framework, which accounts for restrained warping of the cross section and large rotation effects, respectively. Shear deformation effects are included by applying the Timoshenko-Ehrenfest beam theory for bending and a modified Vlasov theory for torsion. A consistent mass matrix of a beam element is derived using a kinetic-energy-based approach, accounting for coupling between translational, rotational, and warping degrees of freedom. The governing equations of motion of the beam element are formulated within the framework of Hamilton's variational principle, leading to the corresponding eigenvalue problem. The proposed formulation is verified through selected examples, demonstrating its effectiveness in predicting the natural frequencies of geometrically nonlinear, thin-walled beam-type structures under initial loading.

Keywords: composite thin-walled cross-section, unbalanced laminates, consistent mass matrix, natural frequencies, initial load, beam model.

1 Introduction

Thin walled structural members made of composite materials are increasingly utilized in a wide range of engineering applications. These structures offer several important advantages, including high stiffness to weight and strength to weight ratios, high load carrying capacity, cost efficiency, and excellent resistance to corrosion. However, their structural response is inherently complex due to pronounced coupling effects and the interaction between warping and torsion. Consequently, accurate prediction of stability limits, particularly buckling resistance, as well as dynamic characteristics is essential for the reliable and optimal design of composite structures. In recent years, considerable research efforts have been devoted to the development of analytical and numerical methods for investigating the structural behaviour of thin walled composite members. Although analytical solutions can be obtained for relatively simple configurations, the analysis of structures with complex geometries, boundary conditions, and loading cases generally requires the application of numerical techniques.

Shear deformations play an important role in the structural response of composite members, significantly influencing transverse displacements, natural vibration frequencies, and critical buckling loads. Analyses based on the Euler–Bernoulli beam assumption may lead to considerable inaccuracies when shear deformation effects are neglected [1,2]. To address this limitation, several geometrically nonlinear formulations for composite beam structures that explicitly include shear deformation have been proposed in the literature [3-11]. A substantial number of studies have addressed the modelling of the dynamic behaviour of thin walled laminated beams. Several representative contributions are briefly reviewed here. Cortínez and Piovan [3] developed a theoretical model for the vibration and buckling analysis of shear deformable thin walled composite beams, demonstrating the significant influence of shear deformation on the structural response. Machado and Cortínez [4] examined the free vibration behaviour of thin walled composite beams subjected to static initial stresses and deformations, showing that shear deformability and restrained warping strongly affect natural frequencies and mode shapes. Prokić [12] presented closed form approximate solutions for the flexural–torsional vibration of thin walled composite beams with arbitrary open cross sections. Further developments were reported by Lee and Kim [13,14], who proposed analytical and one dimensional finite element models for open section laminated composite beams, including I and channel shaped cross sections. Vo et al. [15] developed a vibration analysis model for thin walled composite I-beams with arbitrary laminate stacking sequences, which was subsequently used to investigate load–frequency interaction characteristics under different loading conditions. Finally, Vöros [16] proposed a model describing the free vibration behaviour and mode shapes of Bernoulli–Vlasov beams, accounting for bending–torsion coupling induced by steady state lateral loads.

2 Methods

In this formulation, shear deformation effects are incorporated using the Timoshenko-Ehrenfest beam theory for bending and a modified Vlasov theory for nonuniform torsion. Furthermore, the present study introduces an enhanced shear deformable beam formulation that accounts for the coupling between bending-bending and bending-warping torsion associated with shear deformation [6-11]. These coupling effects are particularly significant in asymmetric cross sections, where the principal bending and shear axes do not coincide [8-11]. The beam is assumed to be straight and prismatic, and the external loads are considered conservative and static. The geometric stiffness of the element incorporates a nonlinear displacement field of the cross section [6], which includes second order displacement terms to capture large rotation effects. Consequently, the incremental geometric potential associated with the semitangential moment is evaluated for internal bending and torsional moments. This procedure ensures the equilibrium of moments at frame joints connecting beam members with different spatial orientations [6]. The total kinetic energy of a deformable body with a continuous mass distribution can be expressed as:

$$\mathcal{K} = \frac{1}{2}\int_0^l \int_A \rho\left(\dot{w}^2 + \dot{u}^2 + \dot{v}^2\right)\mathrm{d}A\,\mathrm{d}z \tag{1}$$

where ρ is the material density, A is the cross section area, and l is the length of the thin walled composite beam finite element. A nonlinear displacement field of the thin walled cross section is defined as:

$$\mathbf{u} = \mathbf{u}_0 + \tilde{\boldsymbol{\varphi}}\,\mathbf{r}_0 - \tilde{\boldsymbol{\omega}}\,\mathbf{s} + 0.5\left(\tilde{\boldsymbol{\varphi}}^2\,\mathbf{r}_0 - \tilde{\boldsymbol{\varphi}}\,\tilde{\boldsymbol{\omega}}\,\mathbf{s}\right); \quad \mathbf{u}^{\mathrm{T}} = \{w \quad u \quad v\} \tag{2}$$

where:

$$\mathbf{u}_0 = \begin{Bmatrix} w_{\mathrm{O}} + \Omega\,\theta \\ u_{\mathrm{S}} \\ v_{\mathrm{S}} \end{Bmatrix}; \mathbf{r}_0 = \begin{Bmatrix} 0 \\ x \\ y \end{Bmatrix}; \mathbf{s} = \begin{Bmatrix} 0 \\ x_{\mathrm{S}} \\ y_{\mathrm{S}} \end{Bmatrix}; \tilde{\boldsymbol{\varphi}} = \begin{bmatrix} 0 & -\varphi_{\mathrm{y}} & \varphi_{\mathrm{x}} \\ \varphi_{\mathrm{y}} & 0 & -\varphi_{\mathrm{z}} \\ -\varphi_{\mathrm{x}} & \varphi_{\mathrm{z}} & 0 \end{bmatrix}; \tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\varphi_{\mathrm{z}} \\ 0 & \varphi_{\mathrm{z}} & 0 \end{bmatrix} \tag{3}$$

For dynamic analysis, the corresponding linear velocity field is obtained by taking the time derivative of the displacement field:

$$\dot{\mathbf{u}} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}; \quad \dot{\mathbf{u}}^{\mathrm{T}} = \{\dot{w} \quad \dot{u} \quad \dot{v}\} \tag{4}$$

Substituting the velocity field from Equation (4) into Equation (1) and performing the required simplifications yields the following expression for the kinetic energy:

$$\begin{aligned}\mathcal{K} = \frac{1}{2}\int_0^l \Big[&\rho A(\dot{w}_{\mathrm{O}}^2 + \dot{u}_{\mathrm{S}}^2 + \dot{v}_{\mathrm{S}}^2) + \rho I_{\mathrm{pS}}\dot{\varphi}_z^2 + \rho I_{\mathrm{x}}\dot{\varphi}_x^2 + \rho I_{\mathrm{y}}\dot{\varphi}_y^2 + \rho I_{\omega}\dot{\theta}^2 + \\ &+2\rho A\,y_{\mathrm{S}}\,\dot{u}_{\mathrm{S}}\,\dot{\varphi}_z - 2\rho A\,x_{\mathrm{S}}\,\dot{v}_{\mathrm{S}}\,\dot{\varphi}_z + 2\rho S_x \dot{w}_{\mathrm{O}}\,\dot{\varphi}_x - 2\rho S_y\,\dot{w}_{\mathrm{O}}\,\dot{\varphi}_y + 2\rho S_{\omega}\,\dot{w}_{\mathrm{O}}\,\dot{\theta} - \\ &-2\rho S_x \dot{u}_{\mathrm{S}}\,\dot{\varphi}_z + 2\rho S_y\,\dot{v}_{\mathrm{S}}\,\dot{\varphi}_z - 2\rho I_{xy}\,\dot{\varphi}_x\dot{\varphi}_y + 2\rho I_{y\omega}\dot{\varphi}_x\dot{\theta} - 2\rho I_{x\omega}\dot{\varphi}_y\,\dot{\theta}\Big]\mathrm{d}z\end{aligned} \tag{5}$$

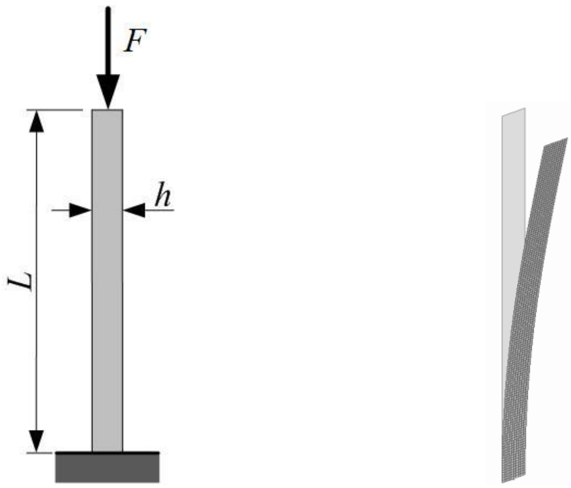
where the properties of the cross section are defined as follows:

$$\begin{aligned}&A = \int_A \mathrm{d}A;\ I_x = \int_A y^2\mathrm{d}A;\ I_y = \int_A x^2\mathrm{d}A;\ I_\omega = \int_A \Omega^2\mathrm{d}A;\ I_{p\mathrm{S}} = \int_A \left[(x - x_{\mathrm{S}})^2 + (y - y_{\mathrm{S}})^2\right]\mathrm{d}A; \\ &I_{xy} = \int_A xy\mathrm{d}A;\ I_{x\omega} = \int_A x\Omega\,\mathrm{d}A;\ I_{y\omega} = \int_A y\Omega\,\mathrm{d}A;\ S_x = \int_A y\mathrm{d}A;\ S_y = \int_A x\mathrm{d}A;\ S_\omega = \int_A \Omega\,\mathrm{d}A\end{aligned} \tag{6}$$

In Equation (3), w_{O}, u_{S} and v_{S} denote the rigid body translational increments of the cross section, φ_z, φ_x and φ_y represent the rigid body rotational increments of the cross section, θ is the warping parameter of the cross section, Ω is the principal sectorial coordinate or warping function. The nonlinear displacement field captures large deformation effects and geometric nonlinearities, ensuring accurate representation of the beam kinematics.

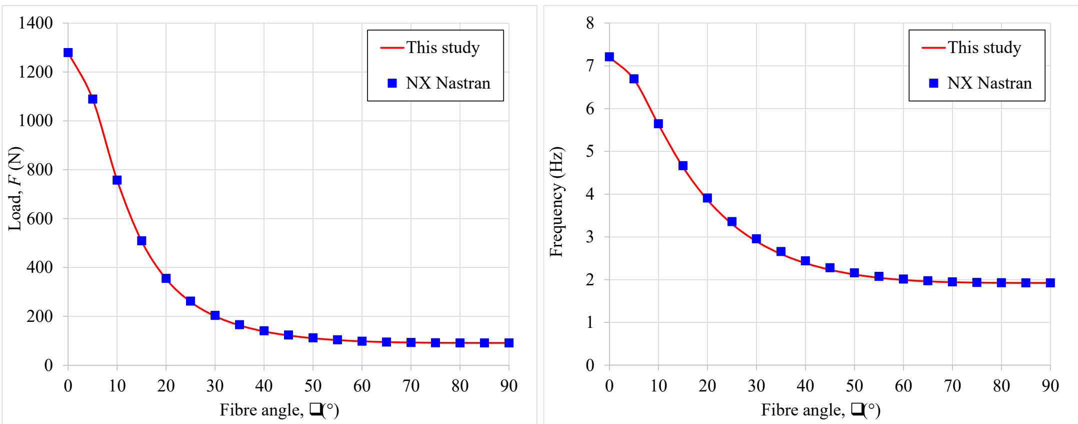
3 Results

The finite element formulation described above was implemented in a computer program called EIGEN. The program is designed to perform both buckling and vibration analyses of thin-walled composite beam-type structures.



(a) (b)

Figure 1: Cantilever column: (a) geometry; (b) natural mode.

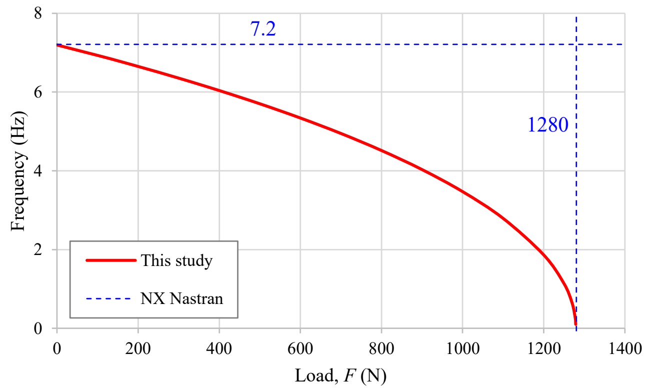


(a) (b)

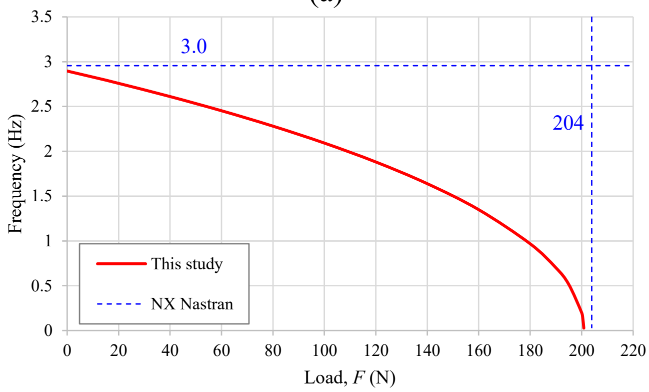
Figure 2: Cantilever column: (a) buckling load vs. fibre angle; (b) first natural frequency vs. fibre angle.

In the benchmark example presented for verification, the material considered is graphite-epoxy (AS4/3501), characterized by longitudinal and transverse elastic moduli of E_1=140 GPa and E_2=10 GPa, respectively, a shear modulus of G_{21}=4.14

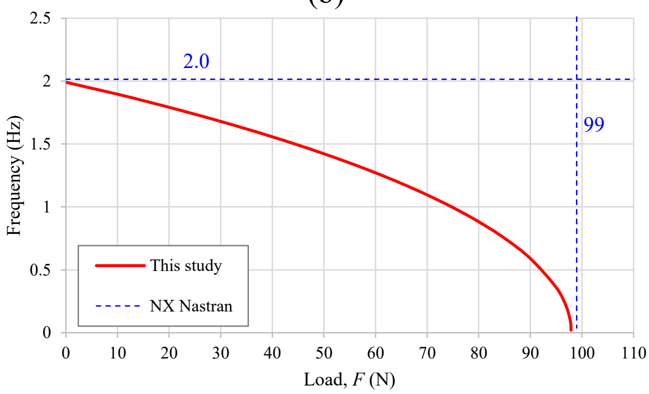
GPa, Poisson's ratio ν_{12}=0.3, and a density ρ = 1389 kg/m^3. Cantilever column, shown in Figure 1, is considered. The column has a length of L=150 cm and a rectangular cross section of $t \times h = 1 \times 10$ cm. The laminate stacking sequence is $[\theta]_4$.



(a)



(b)



(c)

Figure 3: Cantilever column, first natural frequency vs. load: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 60°$.

In Figure 2, the variations of the buckling load and the first natural frequency as functions of the fibre angle are presented. These results illustrate the influence of fibre orientation on both the stability and the dynamic behavior of the composite structure. By varying the fibre angle, the stiffness characteristics of the laminate change, which directly affects the critical buckling load and the natural frequencies of the column.

The obtained results are compared with those calculated using a shell finite element model composed of 1500 laminate shell elements. This model represents a more detailed numerical discretization of the structure and is used as a reference solution for validation purposes. In both cases, namely for the buckling load and for the first natural frequency, the results show very good agreement with those obtained using the laminate shell model implemented in NX Nastran.

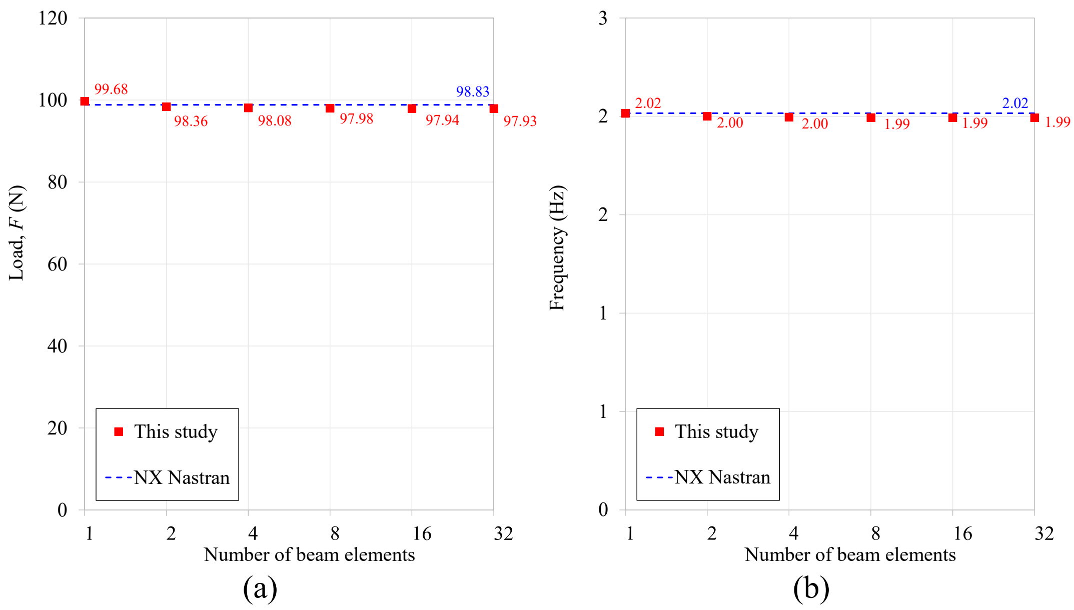


Figure 4: Cantilever column, $\theta = 60°$: (a) buckling load vs. number of beam elements; (b) first natural frequency vs. number of beam elements.

Figure 3 illustrates the influence of the applied compressive load on the value of the first natural frequency. As the compressive load increases, the stiffness of the structure effectively decreases, which results in a gradual reduction of the first natural frequency. When the compressive load reaches the critical buckling value, the first natural frequency approaches zero, indicating the loss of structural stability. This relationship provides an alternative way to determine the critical buckling load, since the buckling force corresponds to the load level at which the first natural frequency becomes zero. The results presented in Figure 3 show good agreement with those obtained using the laminate shell model implemented in NX Nastran, confirming the validity and accuracy of the developed formulation.

A convergence study, presented in Figure 4, evaluates six different mesh configurations consisting of 1, 2, 4, 8, 16, and 32 beam elements. In this analysis, a fibre angle of $\theta = 60°$ is considered. The purpose of the study is to examine the influence of mesh refinement on the accuracy and stability of the obtained results. The results show that the solution converges rapidly with increasing mesh density. Even the model consisting of a single beam element provides stable and reasonably accurate results, indicating that the proposed formulation is numerically efficient and capable of capturing the essential structural behavior with a relatively coarse discretization.

4 Conclusions and Contributions

This paper presents an enhanced beam model for predicting the natural frequencies of thin-walled composite beams and frame structures. Each wall of the cross-section is modeled as a thin, symmetric, and unbalanced angle-ply laminate. The proposed approach accounts for geometrically nonlinear large-displacement effects, shear deformation, and cross-sectional warping. A consistent mass matrix is derived using a kinetic-energy-based formulation, capturing the coupling between translational, rotational, and warping degrees of freedom. The natural frequencies of the structure are then obtained by solving the corresponding eigenvalue problem. The benchmark example demonstrate the accuracy and robustness of the proposed model. In particular, the example include torsional–flexural vibrations of a column. The numerical results show excellent agreement with the solutions obtained using a laminate shell model implemented in NX Nastran. Overall, the proposed formulation provides a reliable and efficient tool for predicting the natural frequencies of thin-walled composite beam structures with unbalanced and symmetric laminate configurations subjected to initial loading conditions.

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