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An Efficient Multiscale Topology Optimization Framework Using HiCon-FEM

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Abstract

This paper presents a multiscale topology optimization framework that combines the classical Solid Isotropic Material with Penalization (SIMP) method with the Hierarchical Condensation Finite Element Method (HiCon-FEM). The aim is to preserve the simplicity and robustness of density-based topology optimization while reducing the computational cost associated with the repeated solution of large finite element systems. The formulation is solved with an Optimality Criteria update scheme, and a Gaussian convolution-based filter is employed as a scalable regularization strategy for large meshes. The main contribution is the use of HiCon-FEM as a drop-in replacement for the standard finite element solver. Numerical results for the MBB beam benchmark show speedups of up to $40.0\times$ for a 900×300 discretization and up to $35.0\times$ for a 3000×2000 discretization while preserving the main topological features of the classical SIMP solutions.

Keywords: HiCon-FEM, SIMP, topology optimization, hierarchical condensation, multiscale optimization, Gaussian filter

1 Introduction

Topology optimization is a computational methodology for distributing material inside a prescribed domain in order to optimize structural performance under loads, supports, and resource constraints. In structural mechanics, the most common objective is compliance minimization, which seeks the stiffest layout for a prescribed volume fraction. The historical roots of the field go back to Michell's classic work on optimal truss layouts [13], but modern computational topology optimization began to consolidate with homogenization-based and density-based formulations [6, 5, 7]. A crucial milestone was the realization that optimal continua often correspond to spatially varying porous microstructures, which naturally opened the door to multiscale thinking.

Among the available formulations, the Solid Isotropic Material with Penalization method became the dominant framework because of its conceptual simplicity, numerical robustness, and compatibility with gradient-based optimization [7, 14]. In SIMP, each element is assigned a pseudo-density between zero and one, and the material stiffness is interpolated through a penalized law that discourages intermediate densities. This makes the method straightforward to implement on fixed meshes while still producing near-binary designs when combined with suitable regularization and continuation strategies.

Despite its success, the main limitation of SIMP remains computational. Each iteration of the optimization loop requires at least one large finite element analysis, which becomes especially burdensome in high-resolution settings [1], and in high-resolution problems this analysis dominates the total runtime. This bottleneck becomes even more severe in multiscale settings, where macro-level design updates are coupled to microstructural resolution. The present paper addresses this limitation by integrating HiCon-FEM into the topology optimization loop. Rather than altering the mathematical structure of SIMP, the proposed framework replaces the classical finite element solver with a reduced-order alternative that preserves the same inputs and outputs while dramatically lowering the cost of structural analysis.

2 Methodology

The broad family of topology optimization methods can be organized according to the way the material layout is parameterized and evolved. Density-based methods represent the structure through continuous element-wise variables and are especially effective for compliance minimization. Level-set methods describe the solid–void interface as the zero contour of an implicit function, which naturally yields crisp boundaries and facilitates the imposition of geometric constraints [2, 3]. Phase-field methods introduce a diffused interface governed by gradient regularization and double-well potentials, providing a rigorous variational route to smooth and nearly binary topologies [17, 4, 10]. Evolutionary methods, such as ESO and BESO, adopt a heuristic strategy by removing or reintroducing inefficient material according to local performance indicators [19, 20].

Although these formulations differ in implementation details and numerical behavior, they all seek to resolve the same fundamental design question: where should material be placed to maximize structural efficiency? In practice, density-based SIMP remains the preferred option in many engineering contexts because it strikes an excellent balance between mathematical tractability, implementation effort, and computational efficiency. For this reason, the present work adopts SIMP as the optimization backbone and concentrates its innovation on the finite element stage, where the computational burden is most acute.

Modern research has also pushed topology optimization far beyond its original structural setting. Thermal, acoustic, fluidic, vibro-acoustic, and electromagnetic applications have all been formulated within a topology optimization perspective [15]. At the same time, additive manufacturing has stimulated a renewed interest in hierarchical and multiscale optimization, since contemporary fabrication technologies can realize graded microstructures, lattice infills, and architected metamaterials that were previously out of reach [18, 12]. These developments make efficient multiscale solvers especially valuable, because the practical use of complex optimized architectures depends not only on design freedom but also on the ability to analyze them repeatedly within an iterative optimization process.

2.1 SIMP formulation and Optimality Criteria update

Let the design domain be discretized into N_e finite elements, each associated with a pseudo-density $\rho_e \in [0,1]$. In the classical SIMP framework, the compliance minimization problem under equilibrium and a global volume constraint is written as [7]

$$\begin{aligned} \min_{\boldsymbol{\rho}} \quad & C(\boldsymbol{\rho}) = \mathbf{f}^{\mathrm{T}}\mathbf{u} = \mathbf{u}^{\mathrm{T}}\mathbf{K}(\boldsymbol{\rho})\mathbf{u} \\ \text{s.t.} \quad & \mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f}, \\ & \frac{1}{N_e}\sum_{e=1}^{N_e} \rho_e \leq \bar{\rho}, \\ & 0 < \rho_{\min} \leq \rho_e \leq 1, \qquad e = 1,\dots,N_e, \end{aligned} \tag{1}$$

where $\mathbf{u}$ is the global displacement vector, $\mathbf{f}$ is the external load vector, $\mathbf{K}(\boldsymbol{\rho})$ is the global stiffness matrix, and $\bar{\rho}$ is the prescribed volume fraction. In the SIMP model, the elemental stiffness is interpolated according to

$$\mathbf{K}_e(\rho_e) = \rho_e^p \mathbf{K}_0, \tag{2}$$

where $\mathbf{K}_0$ is the stiffness matrix of the solid element and p is the penalization exponent.

The appeal of SIMP lies in the fact that the design variables are continuous, which permits the use of efficient gradient-based optimization schemes. In the present work, the densities are updated by means of the Optimality Criteria (OC) method [7, 14]. For compliance minimization, the sensitivity of the objective function with respect to each density variable is obtained from the elemental strain energy, yielding

$$\frac{\partial C}{\partial \rho_e} = -p\, \rho_e^{p-1}\, \mathbf{u}_e^{\mathrm{T}} \mathbf{K}_0 \mathbf{u}_e, \tag{3}$$

where $\mathbf{u}_e$ denotes the elemental displacement vector. For uniform elements, the derivative of the volume constraint with respect to ρ_e is constant.

The OC update is obtained by introducing a Lagrange multiplier λ associated with the volume constraint and enforcing the corresponding optimality condition. In practice, the design variables are updated through

$$\rho_e^{\text{new}} = \max(\rho_{\min}, \max(\rho_e - m, \min(1, \min(\rho_e + m,\ \rho_e\, B_e^{\eta})))) \,, \tag{4}$$

with

$$B_e = -\frac{1}{\lambda}\, \frac{\partial C/\partial \rho_e}{\partial V/\partial \rho_e}, \tag{5}$$

where m is the move limit and η is a damping parameter. The multiplier λ is determined by a bisection procedure so that the prescribed volume fraction is satisfied at each iteration.

These sensitivities provide the local information required by the OC rule and, once filtered, drive the iterative evolution of the design. Since the overall optimization framework is classical and well established, any meaningful reduction in the cost of the finite element stage translates directly into an acceleration of the complete topology optimization process.

2.2 Numerical regularization and Gaussian filtering

A naïve density-based formulation suffers from numerical instabilities, the most prominent being checkerboard patterns and mesh dependence [16]. These artifacts arise from the discrete nature of the finite element mesh and produce non-physical, highly oscillatory layouts that are not representative of meaningful structural optima. Regularization is therefore indispensable in practical topology optimization. Figure 1 illustrates the effect of Gaussian smoothing within the optimization loop.

The traditional sensitivity filter replaces each raw sensitivity by a weighted average of neighboring values inside a prescribed radius [8]. This introduces a minimum length scale and suppresses high-frequency oscillations. However, for very large meshes, explicitly storing and applying neighborhood-based weights can become memory intensive. To address this issue, the framework considered here employs a Gaussian convolution-based filter. Instead of a linearly decaying hat kernel,

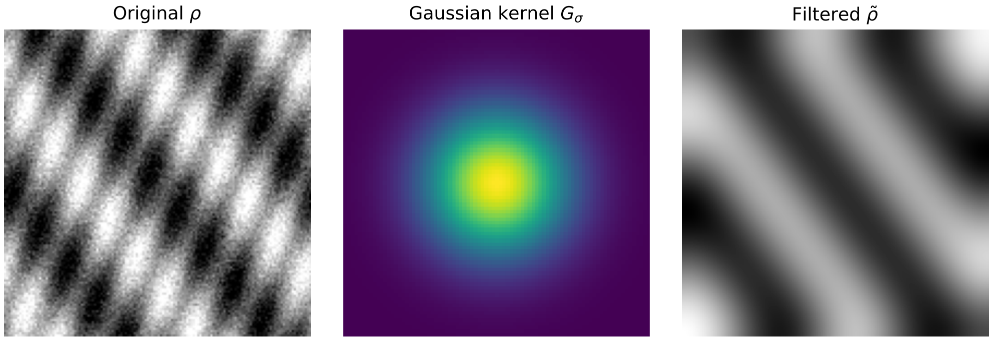


Figure 1: Two-dimensional Gaussian filtering of a noisy density field. (a) Original $\rho(x, y)$, (b) Gaussian kernel G_σ, and (c) filtered field $\tilde{\rho}(x, y)$. The convolution smooths local variations while preserving the overall material distribution, illustrating the spatial regularization achieved by the Gaussian filter.

the sensitivities are smoothed by a radially symmetric Gaussian kernel whose influence decreases smoothly with distance.

The Gaussian kernel can be implemented efficiently through convolution, avoiding the explicit construction of large sparse weight matrices. From a regularization standpoint, this plays a role analogous to more classical filter-based stabilization strategies [8, 9]. This makes the procedure particularly attractive when the design domain contains millions of elements. In the context of the present work, Gaussian filtering is not an isolated numerical trick but a complementary ingredient that helps HiCon-FEM scale to very large discretizations while maintaining stable design evolution. The effect of the Gaussian filter is illustrated in Figure 1, where a noisy density field is smoothed through convolution with a Gaussian kernel while preserving the overall material distribution.

2.3 Integration of HiCon-FEM into the optimization loop

The proposed strategy integrates HiCon-FEM into a standard density-based topology optimization framework by replacing the repeated full-order finite element analyses with a reduced-order solver acting at the macroelement level. The design domain is partitioned into macroelements, each containing a fine underlying microstructural discretization. In this sense, the approach remains connected to the broader tradition of multiscale and homogenization-inspired topology optimization [11, 12], but avoids repeated full-resolution microstructural solves by introducing a hierarchical approximation and static condensation procedure. The overall workflow is illustrated in Figure 2.

For each macroelement, let $\mathbf{K}_e$ denote the local stiffness matrix associated with the underlying microstructural discretization, and let $\mathbf{A}_e$ be the hierarchical interpolation matrix that maps reduced modal degrees of freedom to the micro-scale kinematic field.

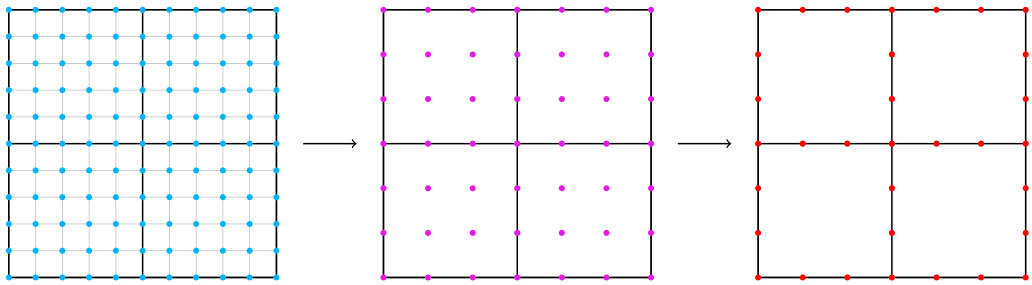


Figure 2: Schematic representation of the integration of HiCon-FEM into the topology optimization loop. The density field is mapped to macroelements, local microstructural problems are projected onto a hierarchical basis and condensed, the reduced global system is assembled and solved, and the local fields are reconstructed for objective and sensitivity evaluation.

The local full-order problem is first projected onto the hierarchical space according to

$$\mathbf{K}_e^r = \mathbf{A}_e^{\mathrm{T}} \mathbf{K}_e \mathbf{A}_e, \qquad \mathbf{f}_e^r = \mathbf{A}_e^{\mathrm{T}} \mathbf{f}_e, \tag{6}$$

where $\mathbf{K}_e^r$ and $\mathbf{f}_e^r$ are the reduced stiffness matrix and reduced load vector, respectively. This projection replaces the original microstructural displacement field by a reduced hierarchical representation, thereby compressing the local problem before any global coupling is considered.

The reduced local system is then partitioned into boundary (or external) and internal modal degrees of freedom,

$$\mathbf{K}_e^r = \begin{bmatrix} \mathbf{K}_{ee} & \mathbf{K}_{ei} \\ \mathbf{K}_{ie} & \mathbf{K}_{ii} \end{bmatrix}, \qquad \mathbf{u}_e^r = \begin{bmatrix} \mathbf{u}_e \\ \mathbf{u}_i \end{bmatrix}, \qquad \mathbf{f}_e^r = \begin{bmatrix} \mathbf{f}_e \\ \mathbf{f}_i \end{bmatrix}. \tag{7}$$

Static condensation is then applied to eliminate the internal modes, yielding the condensed operator

$$\mathbf{K}_e^{\mathrm{cond}} = \mathbf{K}_{ee} - \mathbf{K}_{ei} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ie}, \tag{8}$$

and, when required, the associated condensed right-hand side

$$\mathbf{f}_e^{\mathrm{cond}} = \mathbf{f}_e - \mathbf{K}_{ei} \mathbf{K}_{ii}^{-1} \mathbf{f}_i. \tag{9}$$

As a result, each macroelement contributes only through the reduced set of external modes that participate in inter-element coupling.

Once all macroelements have been reduced in this way, the global condensed system is assembled in the standard finite element manner,

$$\mathbf{K}^{\mathrm{cond}} \mathbf{u}^{\mathrm{cond}} = \mathbf{f}^{\mathrm{cond}}, \tag{10}$$

where $\mathbf{K}^{\text{cond}}$ is now substantially smaller than the full microstructural system. After solving for the coupled condensed unknowns, the internal modal amplitudes are recovered locally by back-substitution,

$$\mathbf{u}_i = \mathbf{K}_{ii}^{-1} \left(\mathbf{f}_i - \mathbf{K}_{ie} \mathbf{u}_e \right), \tag{11}$$

and the micro-scale displacement field inside each macroelement is reconstructed as

$$\mathbf{u}_e \approx \mathbf{A}_e \mathbf{u}_e^r. \tag{12}$$

This reconstruction provides the displacement information required for the evaluation of compliance and elemental sensitivities within the SIMP loop.

From the viewpoint of the optimizer, the external interface remains unchanged: the solver still receives densities, material parameters, loads, and boundary conditions, and returns displacements together with the local energetic quantities required for sensitivity analysis. The reduction is therefore entirely internal to the structural analysis stage. This modularity is one of the main advantages of the proposed framework, since it allows HiCon-FEM to operate as a drop-in replacement for the classical finite element solver within an otherwise standard topology optimization procedure.

Because the projection and condensation stages are performed independently for each macroelement, the computational work is naturally localized and well suited to parallel execution. This locality, together with the substantial reduction in the size of the global system, is what makes the proposed HiCon-FEM formulation especially attractive for large-scale multiscale topology optimization problems.

3 Numerical results

This section presents the numerical validation and performance assessment of the proposed HiCon-FEM solver within the SIMP topology optimization framework. The objective is to evaluate both the qualitative agreement of the optimized layouts and the computational efficiency achieved with the reduced solver.

3.1 Benchmark setup

The classical MBB beam problem is adopted as the reference benchmark due to its well-established role in topology optimization. The full design domain is optimized without imposing symmetry conditions. The left edge is constrained in the horizontal direction, the bottom-right corner is fixed vertically to prevent rigid-body motion, and a downward point load is applied at the upper-left corner.

All simulations use the same numerical and material parameters in both the classical FEM and HiCon-FEM implementations. The SIMP penalization factor is set to $p = 3.0$, the target volume fraction to $V^* = 0.5$, the Young's modulus to $E_0 = 1$ GPa,

Figure 3: Design domain, load, and boundary conditions of the MBB beam benchmark. The left edge is fixed horizontally, the bottom-right corner vertically, and a downward load acts at the upper-left corner.

and Poisson's ratio to $\nu = 0.3$. A minimum density $\rho_{\min} = 10^{-9}$ is imposed to prevent singular stiffness matrices.

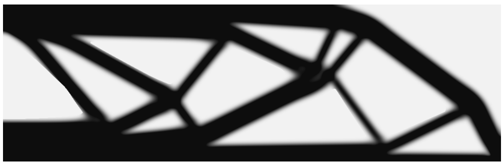
Two benchmark resolutions are considered. In the first case, a 900×300 element mesh is partitioned into 30×10 macroelements, each containing 30×30 microelements. The number of hierarchical approximation modes per macroelement is varied among $m \in \{16, 25, 36\}$ in order to assess the effect of the reduced basis size. In the second case, a 3000×2000 element mesh is studied using two macro discretizations, namely 30×20 and 60×40, both with 36 hierarchical approximation modes per macroelement.

The sensitivity filter radius is chosen as $r_{\min} = 11$ for the 900×300 problem and $r_{\min} = 80$ for the 3000×2000 problem. For the smaller problem, a classical weighted sensitivity filter is employed, whereas for the large-scale problem a Gaussian convolution filter is adopted in order to reduce the memory cost associated with the explicit construction of neighborhood-based filtering operators. The convergence tolerance governing design updates is relaxed to $\texttt{tol} = 0.06$ and $\texttt{tol} = 0.16$, respectively, and all simulations are carried out in Julia and timed using `BenchmarkTools.jl`.

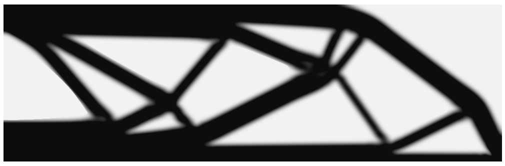
3.2 MBB beam optimization

Figure 4 shows the optimized topologies obtained for the 900×300 design domain. The classical solution is compared against three HiCon-TO configurations using 30×10 macroelements and $m = 16$, $m = 25$, and $m = 36$ hierarchical approximation modes per macroelement. In all cases, the reduced-order solutions reproduce the main structural members and characteristic load paths of the reference design.

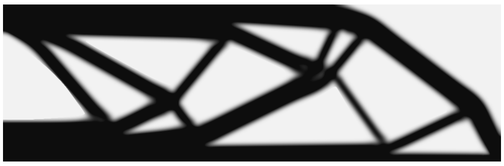
The corresponding performance metrics are reported in Table 1. The results show that HiCon-TO yields substantial acceleration with respect to the classical implementation. Depending on the number of retained approximation modes, the total speedup



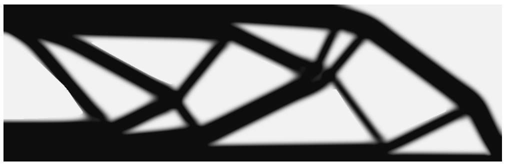
(a) Classical TO



(b) HiCon-TO, $m = 16$



(c) HiCon-TO, $m = 25$



(d) HiCon-TO, $m = 36$

Figure 4: MBB beam topology optimization results for a 900×300 design domain with filtering radius $r_{\min} = 11$.

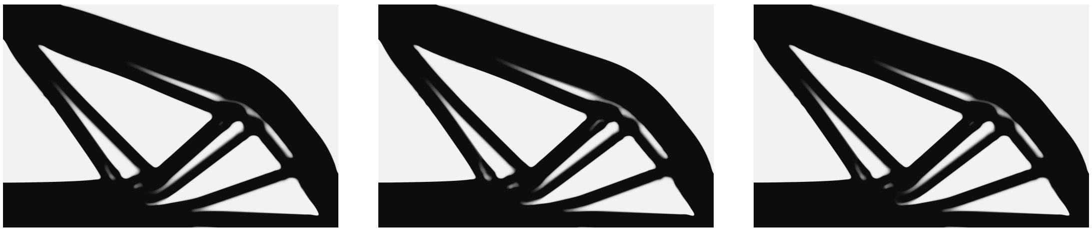
ranges from $15.6\times$ to $40.0\times$. As expected, increasing the number of modes improves the representational richness of the reduced model but also increases the cost of local condensation, revealing a clear trade-off between efficiency and approximation fidelity.

Table 1: Performance and speedup comparison for the MBB beam problem (900×300, $r_{\min} = 11$).

	Classical TO	**HiCon-TO**		
		$m = 16$	$m = 25$	$m = 36$
Objective function	198.997	193.987	195.391	195.979
Execution time [s]	520.70	13.02	14.14	33.47
Number of iterations	162	92	98	169
Average time / iteration [s]	3.21	0.142	0.144	0.198
Total speedup (T_{classic}/T)	$1.0\times$	$40.0\times$	$36.8\times$	$15.6\times$
Total time reduction [%]	0.0	97.5	97.3	93.6
Per-iteration speedup	$1.0\times$	$22.7\times$	$22.2\times$	$16.2\times$
Per-iteration reduction [%]	0.0	95.6	95.5	93.8

For the ultra-high-resolution benchmark, Figure 5 compares the classical topology optimization result with the HiCon-TO solutions obtained using 30×20 and 60×40 macroelement decompositions. Both reduced-order configurations reproduce the global topology with high fidelity, and the differences between the two HiCon-TO solutions are negligible at the structural level.

The quantitative results are summarized in Table 2. In this case, HiCon-TO achieves total speedups of $29.2\times$ and $35.0\times$ for the 30×20 and 60×40 decompositions, respectively. The per-iteration speedups are slightly higher, which indicates that the reduced formulation lowers the cost of each optimization step very significantly. Although the



(a) Classical TO (b) HiCon-TO, 30×20 (c) HiCon-TO, 60×40

Figure 5: MBB beam topology optimization results for a 3000×2000 design domain with Gaussian filtering and $r_{\min} = 80$.

finer macro decomposition increases the size of the reduced global system, it also improves the balance between local condensation and global solution cost, leading here to the best overall runtime.

Table 2: Performance and speedup comparison for the MBB beam problem with 36 approximation modes per macroelement (3000×2000, Gaussian filter, $r_{\min} = 80$).

	Classical TO	**HiCon-TO**	
		30×20	60×40
Objective function	57.987	48.159	50.592
Execution time [s]	25109.03	860.06	716.95
Number of iterations	218	225	225
Average time / iteration [s]	115.18	3.82	3.19
Total speedup (T_{classic}/T)	$1.0\times$	$29.2\times$	$35.0\times$
Total time reduction [%]	0.0	96.6	97.1
Per-iteration speedup	$1.0\times$	$30.1\times$	$36.2\times$
Per-iteration reduction [%]	0.0	96.7	97.2

Taken together, these results show that the proposed HiCon-TO framework preserves the characteristic features of the classical SIMP solutions while delivering substantial reductions in computational cost. For the medium-scale benchmark, the best configuration reaches a total speedup of $40.0\times$, whereas for the ultra-high-resolution problem the achieved speedup reaches $35.0\times$. This confirms that the proposed reduced-order formulation provides a practical route to large-scale multiscale topology optimization.

4 Concluding remarks

This paper presented a multiscale topology optimization framework in which the classical SIMP methodology is combined with the Hierarchical Condensation Finite Element Method (HiCon-FEM). The proposed strategy preserves the standard structure of density-based topology optimization while replacing the repeated full-order finite element analyses with a reduced solver based on hierarchical approximation and static condensation.

The numerical results obtained for the MBB beam benchmark show that the proposed HiCon-TO framework is able to reproduce the main topological features of the classical SIMP solutions while achieving substantial reductions in computational cost. For the 900×300 design domain, the best-performing configuration reached a total speedup of up to $40.0\times$, whereas for the 3000×2000 case the speedup reached $35.0\times$. These results confirm that the finite element stage, which is the dominant computational bottleneck in large-scale topology optimization, can be significantly accelerated without modifying the surrounding optimization machinery.

An additional advantage of the proposed framework is its modularity. Because the reduction is internal to the structural analysis stage, the method can be interpreted as a drop-in replacement for the standard FEM solver within an otherwise classical SIMP loop. This makes the approach attractive not only from a computational perspective, but also from an implementation standpoint, since existing density-based topology optimization codes can be adapted with limited structural changes.

Overall, the results indicate that HiCon-FEM provides a practical and scalable route toward large-scale multiscale topology optimization. Future work may focus on extending the framework to more complex loading scenarios, three-dimensional problems, nonlinear constitutive behavior, and coupled multiphysics settings, where the reduction in computational cost is expected to become even more valuable.

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