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A Finite Element Formulation for Buckling Analysis of Unbalanced Laminated Beam-Type Structures

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Abstract

This work introduces an improved shear-deformable beam formulation buckling analysis of laminated composite beam-type structures with thin-walled cross-sections. Each wall of a cross-section is assumed to be a thin symmetric and unbalanced angle-ply laminate. The equilibrium equations of a straight beam element are derived by applying the virtual work principle within the framework of Lagrangian formulation, Hooke's law and the nonlinear displacement field of a thin-walled cross-section, which takes into account restrained warping and large rotation effects. Stress resultants are calculated by the Timoshenko–Ehrenfest beam theory for bending and the modified Vlasov theories for torsion. Shear coupling problems occurring at unbalanced laminated thin-walled cross-sections and arising from the shear forces-warping torsion moment couplings are considered. The shear-locking occurrence is prevented by applying the Hermitian cubic interpolation functions for deflections and twist rotation, and the associated quadratic functions for slopes and warping. The effectiveness of the proposed geometrically nonlinear shear-deformable beam formulation is validated through the test problems.

Keywords: thin-walled beam-type structure, unbalanced angle-ply laminates, beam finite element, buckling, large rotations, stability analysis.

1 Introduction

Load-bearing composite structures often consist of slender beam elements with thin-walled cross-sections, increasing their complexity and susceptibility to deformation instability or buckling under external loading [1-3]. These optimized structures can exhibit various forms of instability, including pure flexural, pure torsional, torsional-flexural, or lateral deformation. Therefore, accurately determining the buckling strength, the limit state of deformation stability, is crucial in the design process.

While analytical solutions exist for simpler cases [4], numerical approaches are often necessary. Several studies [5-9] have conducted geometric nonlinear analyses of composite beam structures, accounting for shear deformation effects. These studies particularly address bending-bending and bending-warping torsion coupling due to shear deformation, which is especially relevant for asymmetric cross-sections where the principal bending and shear axes do not coincide.

In the previous study by the authors [10], geometrically nonlinear beam elements incorporating shear deformation effects were introduced for composite frames, though the focus was limited to symmetric and balanced laminates. The present work extends this approach by performing a linearized stability analysis of thin-walled beam-type structures, incorporating shear deformation effects and material inhomogeneity in the form of symmetric and unbalanced angle-ply laminates. The analysis is conducted exclusively using the authors' numerical model, with results compared against relevant reference solutions.

2 Methods

This formulation incorporates shear deformation effects by the Timoshenko–Ehrenfest theory for non-uniform bending and the modified Vlasov theory for non-uniform torsion. Additionally, this work introduces an enhanced shear-deformable beam formulation that accounts for bending-bending and bending-warping torsion coupling shear deformation effects [5-11]. These effects become significant in asymmetric cross-sections where the principal bending and shear axes do not coincide [12]. The beam member is assumed to be prismatic and straight, with external loads considered conservative and static.

The geometric stiffness of the element incorporates the nonlinear displacement field of the cross-section [13, 14], which includes second-order displacement terms to account for large rotation effects. As a result, the incremental geometric potential of the semitangential moment is determined for internal bending and torsion moments, ensuring moment equilibrium at frame joints where beam members with different spatial orientations are connected [15, 16].

A locking-free beam element, known as a super-convergent element, is obtained through cubic interpolation for deflections and twist rotation, combined with an interdependent quadratic interpolation for slopes and the warping parameter. This approach effectively incorporates shear-deformable effects without requiring reduced integration techniques to prevent shear locking [17].

To determine properties of a laminated thin-walled cross-section, a particular numerical model is developed. The cross-sectional properties are weighted by using three distinct reference moduli [12]: the longitudinal modulus $\bar{Q}_{11\mathrm{R}}$ governing properties related to the coupling between normal stress forces and normal strains, including area, moments of inertia, area moments, sectorial inertial moments, and sectorial area moments; the shear modulus $\bar{Q}_{66\mathrm{R}}$ governing the torsional moment of inertia and shear coefficients; and the coupling modulus $\bar{Q}_{16\mathrm{R}}$ which governs the coupling between the normal and shear strains. All cross-sectional properties are defined with reference to the midline of each branch of the cross-section.

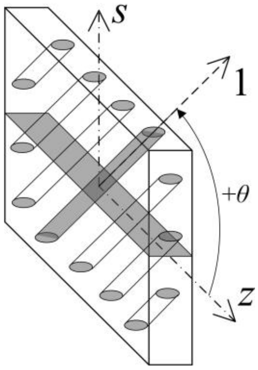


Figure 1: Lamina of an arbitrary orientation.

It should be noted that θ represents the angle between the longitudinal axis (z-axis in this study) of the beam and the fibre direction in a particular ply, as shown in Figure 1. A plane stress condition is assumed for all plies, i.e.

$$\begin{Bmatrix} \sigma_z \\ \tau_{zs} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{16} \\ \bar{Q}_{16} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_z \\ \gamma_{zs} \end{Bmatrix} \tag{1}$$

where:

$$\bar{Q}_{11} = \hat{Q}_{11} - \frac{\hat{Q}_{12}^2}{\hat{Q}_{22}}, \quad \bar{Q}_{66} = \hat{Q}_{66} - \frac{\hat{Q}_{26}^2}{\hat{Q}_{22}}, \quad \bar{Q}_{16} = \hat{Q}_{16} - \frac{\hat{Q}_{12}\hat{Q}_{26}}{\hat{Q}_{22}} \tag{2}$$

while $\hat{Q}_{11}, \hat{Q}_{22}, \hat{Q}_{66}, \hat{Q}_{12}, \hat{Q}_{16}$ and $\hat{Q}_{26}$ are the so-called transformed reduced stiffnesses [18]. For more details on the numerical procedure applied in this work are given in the author's previous papers [9, 10].

3 Results

The abovementioned finite element formulation was implemented in a computer program called EIGEN v.9. The program has capabilities to deal with linearised stability problems of unbalanced symmetric laminated beam-type structures using the eigenvalue approach. The material used in the examples presented afterwards for verification is graphite-epoxy (AS4/3501), characterized by the longitudinal and transverse elastic moduli E_1=144 GPa and E_2=9.65 GPa, respectively, the shear modulus G_{21}=4.14 GPa, and Poisson's ratio ν_{12}=0.3. In the first example, buckling of a cantilever column from Figure 2, of length L=150 cm and made of a rectangular cross-section $t \times h = 1 \times 10$ cm, is analysed. In this, two material configurations are considered: balanced and unbalanced symmetric laminate ones with $[\theta/-\theta]_{2S}$ and $[\theta]_4$ stacking sequences, respectively.

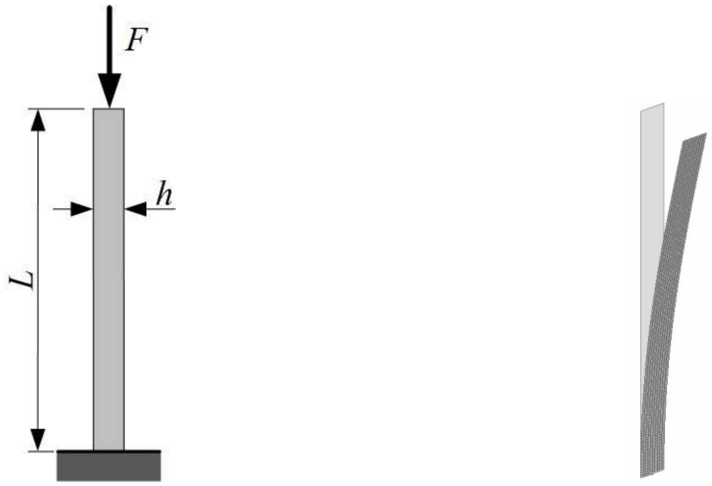


Figure 2: Cantilever column: (a) geometry; (b) buckling mode.

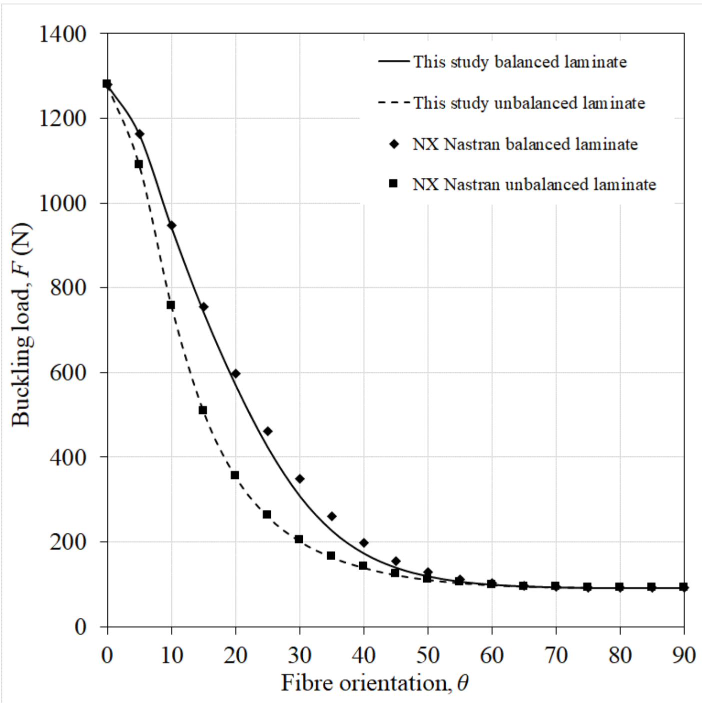
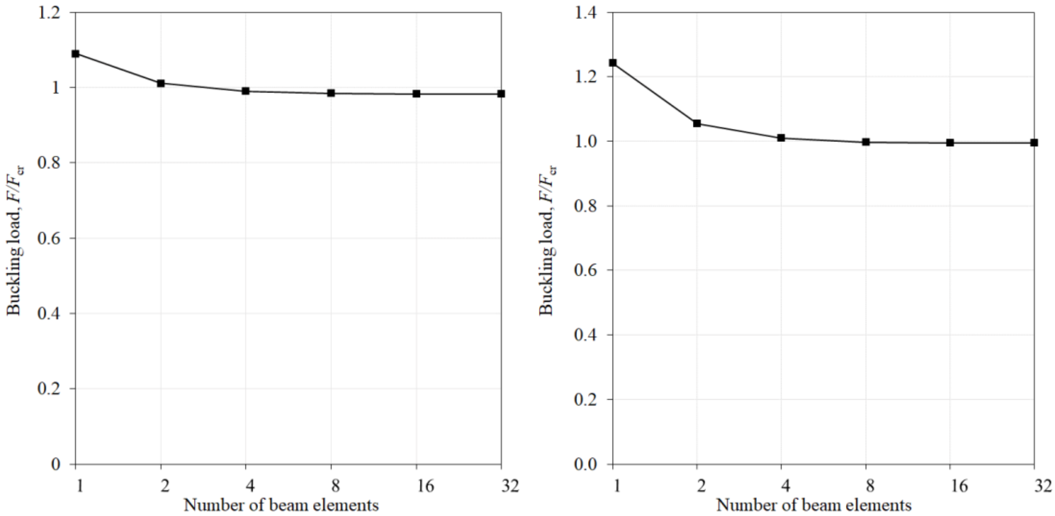


Figure 3: Cantilever column: buckling load versus fibre orientation.

In Figure 3, the variations of the buckling load vs. fibre orientation are presented. Buckling loads are obtained using a mesh configuration consisting of eight beam elements. As it can be seen, the most pronounced difference occurs at $\theta = 15°$. At this fibre orientation, the unbalanced laminate, $[\theta]_4$, gives approximately 35 % lower buckling load value than the balanced one, $[\theta/-\theta]_{2S}$. In both cases, the results are in a good agreement with those obtained by the NX Nastran shell model.



(a) (b)

Figure 4: Cantilever column, buckling load vs. number of beam elements convergence: (a) $[15°/-15°]_{2S}$; (b) $[15°]_4$.

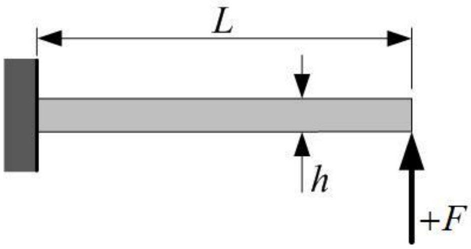
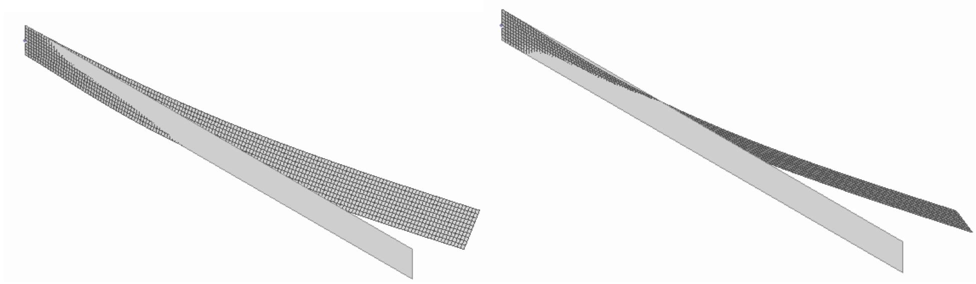


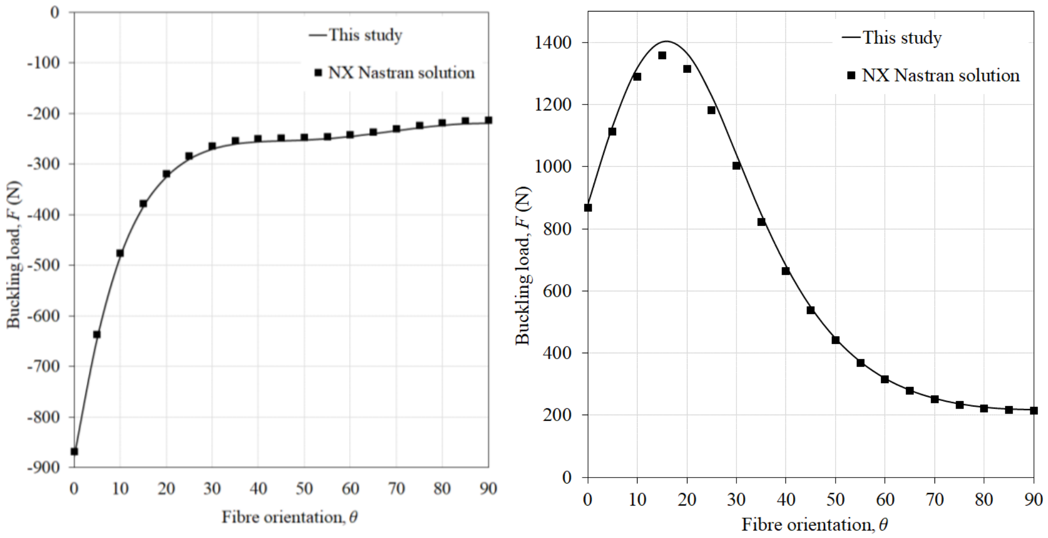
Figure 5: Cantilever beam under lateral load.



(a) (b)

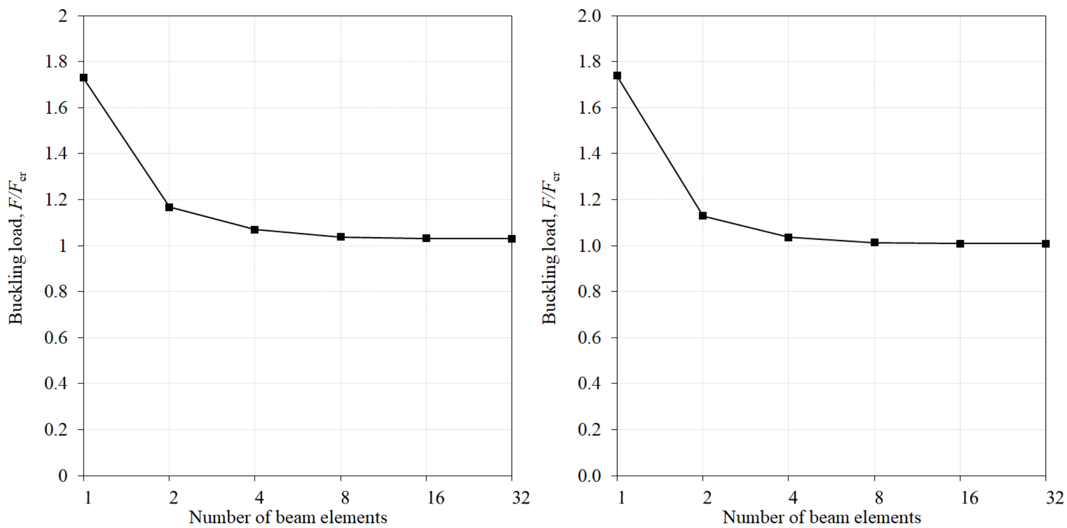
Figure 6: Cantilever beam, lateral-torsional buckling modes: (a) $-F$; (b) $+F$.

A convergent study is carried out as well. The column is idealised using six different mesh configurations each consisting of one, two, four, eight, sixteen and thirty-two beam elements, and the obtained results are shown in Figure 4. In the figure, the obtained buckling load values are normalized by those obtained by the NX Nastran shell model, i.e. F_{cr}=755.14 N and 509.55 N for the balanced and unbalanced cases, respectively.



(a) (b)

Figure 7: Cantilever beam, buckling load vs. ply orientation (θ): (a) $-F$; (b) $+F$.



(a) (b)

Figure 8: Cantilever column, $[15°]_4$ stacking sequence, buckling load vs. number of beam elements convergence: (a) $-F$; (b) $+F$.

In the second example, lateral-torsional buckling of a cantilever beam shown in Figure 5, subjected to a lateral force F is analysed. The beam has the same geometric characteristics as the column from the previous example with the $[\theta]_4$ unbalance stacking sequence. Both force directions are considered, positive and negative, with the positive direction being the one shown in the figure. The corresponding buckling modes obtained by the NX Nastran shell model are shown in Figure 6. The beam is idealised by a mesh configuration consisting of eight beam elements, and the obtained results for the buckling load vs. fibre orientation are shown in Figure 7. As it can be seen, the highest buckling load occurs at θ=0° and 15° for the negative and positive force directions, respectively. As well, the obtained results are in a very good agreement with those obtained by the NX Nastran shell model. The results obtained in the convergence study for are presented in Figure 8, using the same mesh configurations as in the previous example and assuming the fibre orientation θ=15°. The results are normalized by those obtained by the NX Nastran shell model, i.e. F_{cr}=378.04 N and 1359.12 N for the negative and positive force directions, respectively.

4 Conclusions and Contributions

In this study, an improved geometrically nonlinear finite element formulation capable of performing the buckling analysis of shear-deformable beam-type structures, composed of unbalanced angle-ply laminated composites has been presented. Within the frame-work of the virtual work principle and the Lagrangian formulation, and applying the nonlinear displacement field of a thin-walled cross-section, which included the large rotation effects and restrained warping, the equilibrium equations of a straight composite beam element has been derived. Internal stress resultants have been calculated by the Timoshenko–Ehrenfest and modified Vlasov theories for bending and torsion, respectively. Hooke's law has been assumed to be valid. Although displacements have been allowed to be large, strains have been assumed to stay small. To resolve the shear coupling problems occurring at non-symmetric thin-walled cross-sections, an improved shear-deformable beam formulation considering the shear forces-warping torsion moment interactions occurring at beams composed of unbalanced thin angle-ply laminates has been introduced. The reliability of the proposed geometrically nonlinear beam formulation has been verified through two test examples, and the results obtained proved it. The shear-locking testing has also been performed running the model for different mesh configurations.

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