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# Development of 1D Finite Elements With Node-Wise Higher-Order Structural Theories

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## Abstract

This paper presents an innovative modeling approach to increase the accuracy and computational efficiency of Finite Element (FE) models. The present method integrates three key advancements: (1) one-dimensional elements, (2) higher-order structural theories, and (3) a node-dependent kinematics (NDK) framework, where each FE node can possess an independent set of degrees of freedom. Based on the Carrera Unified Formulation (CUF), this framework facilitates the derivation of arbitrary structural theories along with their governing equations and FE arrays. The NDK approach enables spatially varying structural theories. The study focuses on free-vibration analyses of civil engineering beam structures. This work identifies the most efficient spatial distributions of high-fidelity models—those that minimize computational cost while meeting a predefined accuracy threshold. This paper proposes a novel methodology for constructing finite element matrices by dynamically retrieving the active degrees of freedom at each node. The results demonstrate the optimal selection of generalized unknown variables.

**Keywords:** finite element method, Carrera unified formulation, node dependent kinematics, asymptotic-axiomatic method, Taylor polynomials, refined models

# 1 Introduction

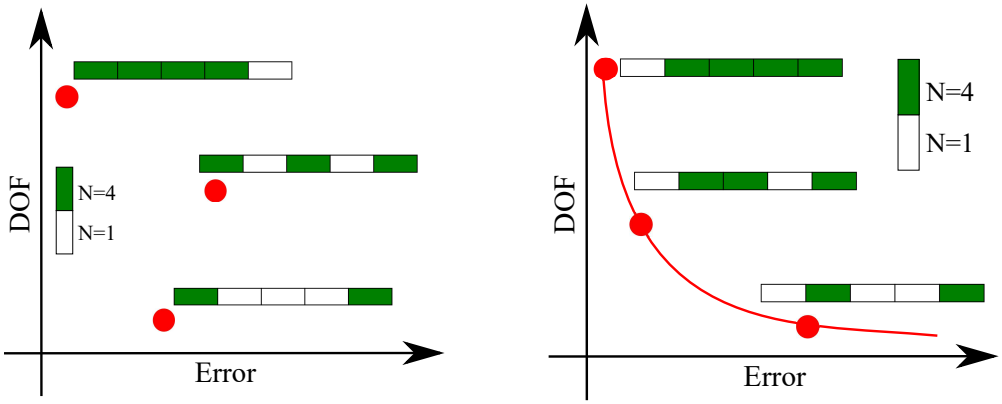
The development of beam theories is a cornerstone of structural engineering, with one-dimensional (1D) models often preferred over more complex three-dimensional analyses. 1D models remain widely used due to their simplicity and efficiency. These models find broad applications across engineering disciplines, especially in civil engineering systems, including metallic and concrete beam frameworks. This section briefly reviews key beam theories.

The Euler-Bernoulli beam model [1] is commonly employed for analyzing slender, isotropic structures but neglects shear deformation. In contrast, the Timoshenko beam theory [2] offers a more comprehensive approach by incorporating shear effects. Finite Element (FE) software typically implements simple one-dimensional elements, each with six degrees of freedom (DOF) per node—three translational and three rotational [3].

Due to the inherent limitations of classical beam theories, researchers have developed more advanced models to address complex structural behaviours. For a comprehensive overview, Novozhilov's seminal work [4] provides valuable insights into these advancements. Additionally, Kapania and Raciti offer a thorough review of various beam theories in their notable study [5]. One of the earliest models designed explicitly for studying thin-walled structures was introduced by Vlasov [6], in which warping functions were incorporated to capture the deformation accurately. This foundational theory has since been applied and extended by numerous scholars, such as Mechab *et al.* [7]. Building upon these efforts, Levinson [8] proposed a beam theory that accounts for warping effects while ensuring shear-free conditions on the lateral surfaces. Wang and Li later employed Levinson's theory to investigate free vibration problems [9].

The selection of an appropriate structural theory is inherently dependent on the specific problem, as the necessity for higher-order generalized variables can vary across the domain. This paper presents a novel methodology for identifying the most suitable structural theories and determining their optimal spatial distribution for the analysis of dynamic issues.

In previous work [10], a penalization technique was employed to identify the most suitable structural theories. In contrast, this paper introduces a novel approach for constructing beam finite element matrices in which each displacement component can be associated with a distinct expansion function at the node level. This enables the integration of various polynomial-based theories within a unified formulation. The proposed methodology leverages the combined strengths of the Carrera Unified Formulation (CUF), the Node-Dependent Kinematics (NDK), and the Axiomatic-Asymptotic Method (AAM). CUF [11] provides the general theoretical framework. NDK [12] permits the selection of different structural theories for each finite element node, offering significant modeling flexibility. AAM [10] is used to assess the accuracy of structural




(a) Three beam NDK theories

(b) Best Theory Distributions

Figure 1: Distributions of structural theories over a 1D mesh (a) and examples of Best Theory Distributions (b)

models across different orders. It generates the Best Theory Diagram (BTD) [13], a two-dimensional plot that correlates structural model accuracy with computational cost.

In a recent study, Petrolo and Carrera [14, 15] employed the AAM and the NDK to determine optimal theory distributions across a two-dimensional mesh. In contrast, the present work extends this methodology to the beam framework. Specifically, the approach enhances AAM by enabling the structural theory to vary at the node level. Figure 1 presents an example of structural theory distributions over a one-dimensional mesh, comparing a linear theory (N=1) with a fourth-order theory (N=4). Figure 1(a) illustrates three different NDK configurations, while Fig. 1(b) depicts a possible Beam Theory Distribution (BTD) obtained from multiple analyses. The structure of this paper is as follows: (a) Section 2 introduces the theoretical foundations of the CUF and NDK; (b) Section 3 derives the governing equations; (c) Section 4 details the AAM with the NDK method; (d) Section 5 presents the numerical results; and (e) Section 6 concludes the study.

## 2 Carrera Unified Formulation

Consider the beam structure illustrated in Fig. 2. In the context of a one-dimensional structural theory, the unknown field variable is denoted as $F_\tau(x,z)$, defined over the cross-sectional area A:

$$f(x,y,z) = F_\tau(x,z) f_\tau(y), \qquad \tau = 1,2,...,M \tag{1}$$

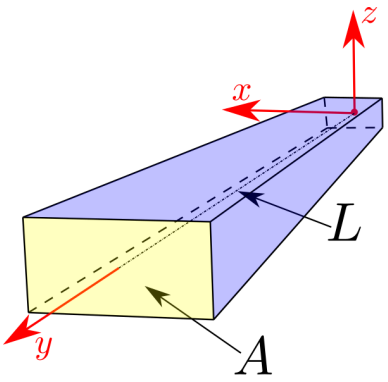



Figure 2: Reference system for a generic beam

Here, Einstein's summation convention is used. The generalized displacement field is expressed as follows:

$$\begin{aligned} u_x(x,y,z) &= F_{u_x\tau}(x,z)u_{x_\tau}(y) \quad \text{with} \quad \tau = 1,\dots,M_{u_x} \\ u_y(x,y,z) &= F_{u_y\tau}(x,z)u_{y_\tau}(y) \quad \text{with} \quad \tau = 1,\dots,M_{u_y} \\ u_z(x,y,z) &= F_{u_z\tau}(x,z)u_{z_\tau}(y) \quad \text{with} \quad \tau = 1,\dots,M_{u_z} \end{aligned} \tag{2}$$

The expansion functions $\mathrm{F}_{u_x\tau}$, $\mathrm{F}_{u_y\tau}$, and $\mathrm{F}_{u_z\tau}$ correspond to the generalized displacements $\mathrm{u}_{x_\tau}$, $\mathrm{u}_{y_\tau}$, and $\mathrm{u}_{z_\tau}$, respectively. The parameters $\mathrm{M}_{u_x}$, $\mathrm{M}_{u_y}$, and $\mathrm{M}_{u_z}$ represent the number of expansion terms associated with each displacement component.

The FEM is employed to discretize displacements along the beam axis. Within this framework, the NDK is introduced, allowing each node to be governed by a distinct structural theory. The NDK also enables local mesh refinement, thereby extending the applicability of this approach. By integrating the capabilities of CUF, FEM, and NDK, the following formulation can be derived:

$$\begin{aligned} u_x(x,y,z) &= N_i(y)F^i_{u_x\tau}(x,z)q_{x_{\tau i}} \quad \text{with} \quad i = 1,\dots,N_n \\ u_y(x,y,z) &= N_i(y)F^i_{u_y\tau}(x,z)q_{y_{\tau i}} \quad \text{with} \quad i = 1,\dots,N_n \\ u_z(x,y,z) &= N_i(y)F^i_{u_z\tau}(x,z)q_{z_{\tau i}} \quad \text{with} \quad i = 1,\dots,N_n \end{aligned} \tag{3}$$

$N_i$ represents the shape functions, where the repeated subscript i implies summation according to Einstein notation. The superscript i on F expansions denotes that each finite element (FE) node can follow a distinct structural theory. $N_n$ refers to the number of nodes per element. In this context, a four-node Lagrange beam element (B4) is employed [3].

# 3 Governing equations and Finite Element matrices

The stress $\boldsymbol{\sigma}$, and strain, $\boldsymbol{\epsilon}$, vectors are defined as follows:

$$\begin{aligned}\boldsymbol{\sigma} &= \left\{ \begin{array}{cccccc} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{yz} & \sigma_{xz} & \sigma_{xy} \end{array} \right\}^T \\ \boldsymbol{\epsilon} &= \left\{ \begin{array}{cccccc} \epsilon_{xx} & \epsilon_{yy} & \epsilon_{zz} & \epsilon_{yz} & \epsilon_{xz} & \epsilon_{xy} \end{array} \right\}^T \end{aligned} \tag{4}$$

The geometric relationships linking strains to displacements are given by:

$$\boldsymbol{\epsilon} = \boldsymbol{D}\boldsymbol{u} \tag{5}$$

The matrix $\boldsymbol{D}$ contains the differential operators and is formulated under the assumption of small displacements and rotations [11]. This work considers linear orthotropic materials:

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon} \tag{6}$$

The matrix $\mathbf{C}$ contains the material coefficients, as detailed in [3].

The governing equations for dynamic analysis are formulated based on the principle of virtual displacements, as follows:

$$\delta L_{int} + \delta L_{ine} = 0 \tag{7}$$

The internal work, $\delta L_{int}$ is

$$\begin{aligned}\delta L_{int} = \int_V \delta\boldsymbol{\epsilon}^T\boldsymbol{\sigma}\mathrm{d}V = \int_V \delta\epsilon_{xx}\sigma_{xx} + \delta\epsilon_{yy}\sigma_{yy} + \delta\epsilon_{zz}\sigma_{zz} \\ + \delta\epsilon_{yz}\sigma_{yz} + \delta\epsilon_{xz}\sigma_{xz} + \delta\epsilon_{xy}\sigma_{xy}\mathrm{d}V\end{aligned} \tag{8}$$

where $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$. The work of inertial forces is

$$\delta L_{kin} = \int_V \rho\delta\boldsymbol{u}^T\ddot{\boldsymbol{u}}\mathrm{d}V = \int_V \rho(\delta u_x\ddot{u}_x + \delta u_y\ddot{u}_y + \delta u_z\ddot{u}_z)\mathrm{d}V \tag{9}$$

By applying the displacement-strain relation, FEM approximation, and constitutive equations, the governing equations for the three displacement components can be obtained,

$$\begin{aligned}\delta q_{x_{sj}} &: K_{u_xu_xs\tau ji}q_{x_{\tau i}} + K_{u_xu_ys\tau ji}q_{y_{\tau i}} + K_{u_xu_zs\tau ji}q_{z_{\tau i}} + M_{u_xu_xs\tau ji}\ddot{q}_{x_{\tau i}} = 0 \\ \delta q_{y_{sj}} &: K_{u_yu_xs\tau ji}q_{x_{\tau i}} + K_{u_yu_ys\tau ji}q_{y_{\tau i}} + K_{u_yu_zs\tau ji}q_{z_{\tau i}} + M_{u_yu_ys\tau ji}\ddot{q}_{y_{\tau i}} = 0 \\ \delta q_{z_{sj}} &: K_{u_zu_xs\tau ji}q_{x_{\tau i}} + K_{u_zu_ys\tau ji}q_{y_{\tau i}} + K_{u_zu_zs\tau ji}q_{z_{\tau i}} + M_{u_zu_zs\tau ji}\ddot{q}_{z_{\tau i}} = 0\end{aligned} \tag{10}$$

The Fundamental Nucleus (FN) of the stiffness and mass matrices consists of scalar quantities, denoted as $\mathrm{K}_{u_mu_ls\tau ji}$ and $\mathrm{M}_{u_mu_ls\tau ji}$, respectively. This formulation enables the theory to be characterized by nine independent scalar parameters for each matrix. The explicit expressions for the fundamental nuclei are not presented here but can be

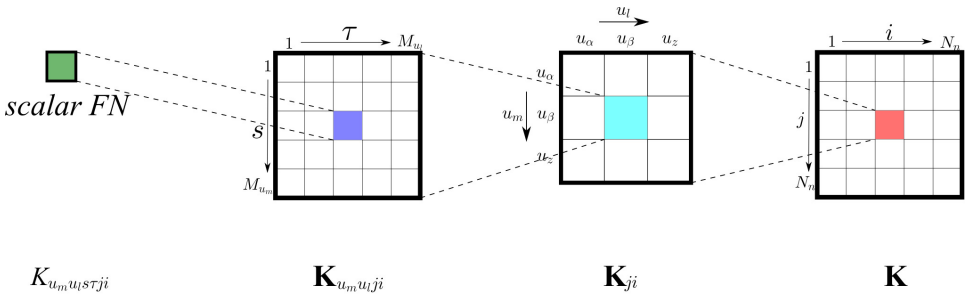



Figure 3: Assembly of the stiffness matrix, starting from the scalar fundamental nucleus.

found in [11, 12]. Assembly over all nodes and elements, combined with the adoption of a harmonic solution, results in the formulation of an eigenvalue problem:

$$\left(-\omega^2 \mathbf{M} + \mathbf{K}\right) \mathbf{U} = \mathbf{0} \tag{11}$$

The assembly process of the stiffness matrix, illustrated in Figure 3, can also be replicated for the mass matrix.

# 4 Asymptotic-Axiomatic Method

The AAM (Automatic Assessment Method) evaluates the accuracy of a given structural theory through the following steps: (a) specifying input data, including geometry, boundary conditions, loadings, and material properties; (b) choosing target output variables (e.g., stresses, displacements, or natural frequencies); (c) selecting finite element mesh; (d) computing the full expansion solution (e.g., all 45 terms active of the fourth-order model) as the benchmark; (e) evaluating different combinations of NDK models, by changing the dispositions of two structural theories; (f) quantifying the accuracy of each reduced model via static or dynamic analyses.

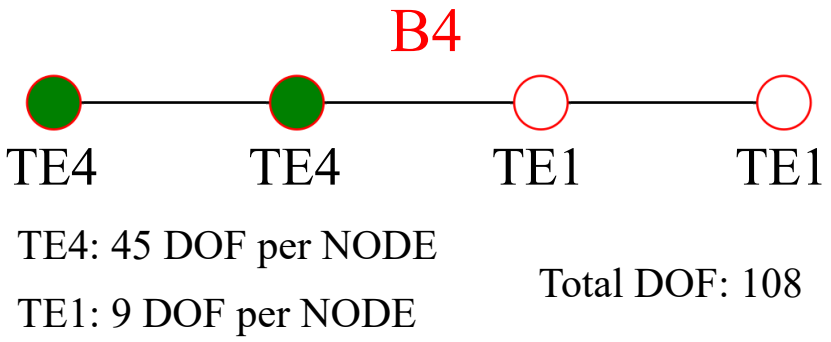



Figure 4: B4 element, a second-order (TE2) beam theory and a reduced model with a truncated expansion

The following example demonstrates the assembly of finite element matrices for an arbitrary reduced structural theory. Consider the four-node element illustrated in

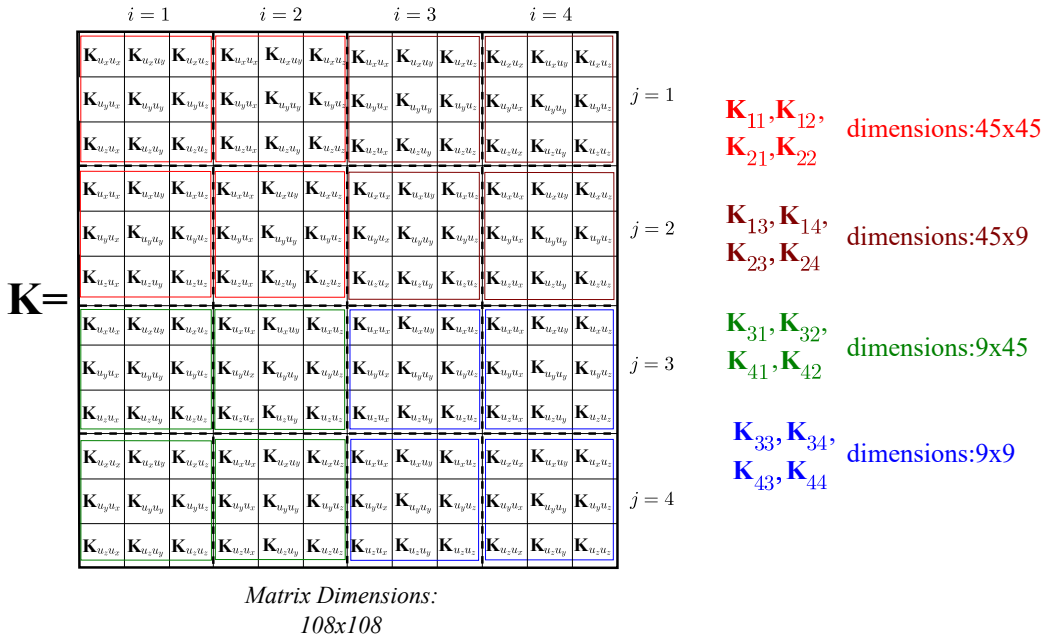



Figure 5: Assembly of the stiffness matrix

Fig. 4. In this configuration, a fourth-order theory (TE4, N = 4) is applied at the first two nodes, while a complete first-order theory (TE1, N = 1) is used at the remaining nodes, resulting in a total of 108 degrees of freedom (DOF). Figure 5 displays the global stiffness matrix $\mathbf{K}$, which has dimensions 108 $times$ 108. For conciseness, the procedure is described only for the stiffness matrix; however, the same methodology applies to the mass matrix $\mathbf{K}$ as well. Each submatrix $\mathbf{K}_{ji}$ is partitioned into nine components $\mathbf{K}_{u_m u_l ji}$, corresponding to the displacement variables. Due to the node-wise variation in structural theory, each submatrix $\mathbf{K}_{ji}$ (or $\mathbf{M}_{ji}$) may have different dimensions, reflecting the differing number of terms at each finite element node. This assembly procedure can consider different structural theories in each node and can lead to the evaluation of best theory distributions over an FE mesh.

# 5 Numerical Results

This section presents the numerical results obtained using the proposed methodology. The benchmark case study, adopted from [16], consists of a simply supported beam with a length-to-height ratio (L/h) of 10. The cross-sectional dimensions are a = 0.38 (m), b = 0.14 (m), and t = 0.02 (m), see Fig. 6. The beam is modelled as an isotropic material with Young's modulus E=72 (GPa), $\nu$=0.33, and $\rho$=2700 (Kg/m$^3$). The first five frequencies (Hz) are written below [16]:
$1^{st}$: 29.46, $2^{nd}$: 66.80, $3^{rd}$: 116.1, $4^{th}$: 253.6, $5^{th}$: 255.2.

Figure 7 show the mesh with ten finite elements. Thus, the total number of the combinations is $2^{10} = 1024$. In this example, only combinations of TE1 and TE4 are

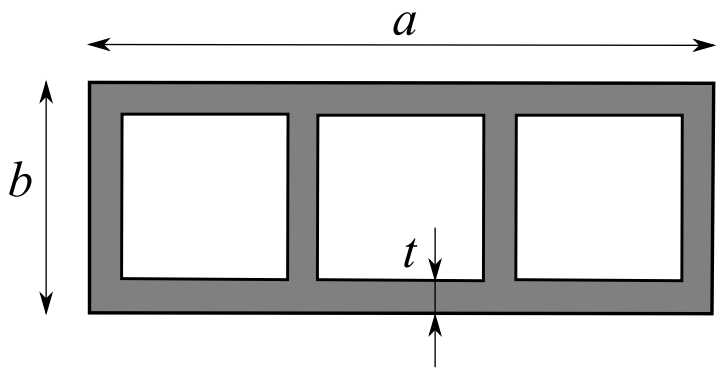
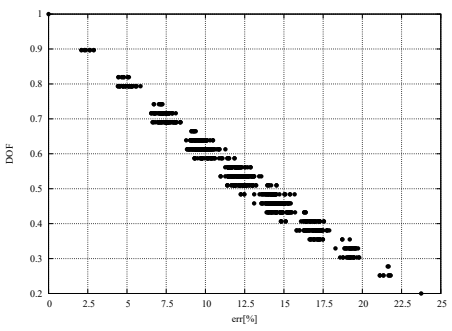


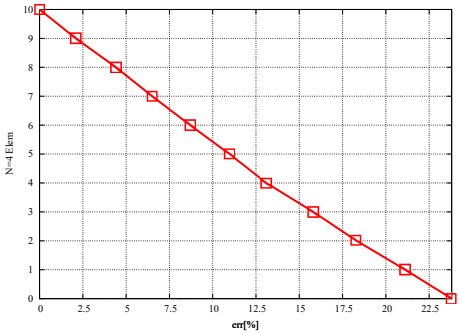

Figure 6: Cross-section of the multi-bay box beam


Figure 7: Finite element scheme for the multi-bay box beam

employed. When two different types of element are employed, the interface nodes use always a first-ored theory. The error is calculated on the first ten frequencies. Figure 8 (a) illustrates the best models for the present method. The boundary values are D=1, corresponding to forty-five DOF per node and 1395 total DOF, and D=0.2, corresponding to nine DOF per node and 279 total DOF. In Fig. 8 (b), the number of elements with a fourth-order theory are represented along the vertical axis. In particular, Fig. 9 illustrates three best models form taken from Fig. 8. In these examples, TE1 theories are used for 2, 5, and 8 elements. The results show that the most critical part is located in the centre of the beam.




(a) All combinations




(b) Best Theory Distributions

Figure 8: NDK results

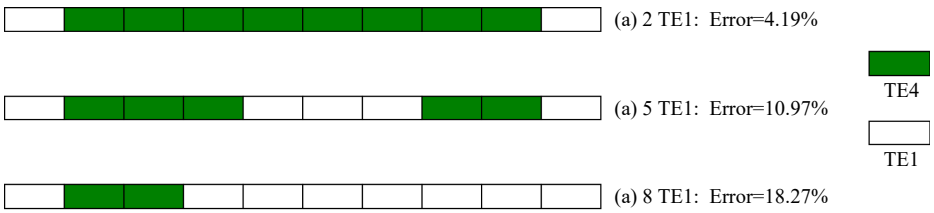



Figure 9: Three best NDK configurations

# 6 Conclusions

This paper introduces an innovative method for selecting optimal models along the beam axis of the structures. Utilizing the enhanced unified formulation, which can generate various theories, the proposed approach allows for a comprehensive study of structures in multiple ways. A multi-box beam structure was analysed using a patch strategy, where each finite element is assigned a structural model (e.g, TE1 or TE4), significantly reducing the number of possible combinations. In this way, the capabilities of the present method have been demonstrated. In future works, this methodology could be coupled with machine learning techniques.

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