Proceedings of the Eighteenth International Conference on
Civil, Structural and Environmental Engineering Computing
Edited by: P. Iványi, J. Kruis and B.H.V. Topping
Civil-Comp Conferences, Volume 10, Paper 8.1
Civil-Comp Press, Edinburgh, United Kingdom, 2025
ISSN: 2753-3239, doi: 10.4203/ccc.10.8.1

On the Discrete Inf-Sup Condition for a Regularized Fictitious Domain Method Using Finite Element Methods

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Abstract

This short paper investigates sufficient conditions for the well-posedness for the discrete version of a regularized fictitious domain method using conforming finite element methods. We show that if both the regularization parameter and the bulk mesh size are proportional to the mesh size of the physical dmain, the discrete *inf-sup* condition holds, where the constant appearing the estimate is independent of all discretization parameters.

Keywords: fictitious domain method, Dirac delta approximations, inf-sup condition, regularization, finite element method, interface problems.

1 Introduction

Non-matching approaches, such as immersed boundary methods [14], cut finite element methods [5, 6], penalty methods [1], and fictitious domain methods [2], are efficient numerical strategies for addressing engineering problems involving complex geometries. These methods utilize a simple computational mesh and allow the boundary of the embedded physical domain to intersect with the mesh, making them particularly useful in shape optimization or free-boundary problems.

In this paper, we focus on the finite element approximation based on the fictitious domain method with a Lagrange multiplier. The corresponding weak formulation (see Section 2 for a formal definition) is formulated as a symmetric saddle-point problem that involves a coupling term between functions defined on the background domain Ω and the physical domain ω:

$$\int_\Gamma \theta(x)v(x)\,\mathrm{d}s_x,$$

where Γ represents the boundary of the physical domain, and θ and v are functions defined on Γ and Ω, respectively. In a finite element setting, evaluating this integrand at quadrature points requires careful consideration when θ and v are finite element basis functions associated with Γ and Ω

One approach involves constructing a fixed quadrature scheme independent of the background mesh. While straightforward to implement, this method introduces errors for nonsmooth functions due to the restriction of v onto Γ. An alternative is to evaluate θ and v at nonzero intersections between cells in the meshes of Γ and Ω. However, this approach is computationally expensive, as it requires (i) computing the intersection between cells in the meshes of Γ and Ω, (ii) constructing a quadrature scheme on these intersections, and (iii) computing the inverse mapping from reference cells to the intersection regions.

To address these challenges, we propose an alternative approach that introduces the Dirac delta function δ and rewrites the coupling integral in terms of the background domain. Then we replace δ with its approximation δ^ε with ε denoting the approximation parameter. Thus we shall instead compute the following double integral

$$\int_\Omega \int_\Gamma \theta(x)\delta^\varepsilon(x-y)v(y)\,\mathrm{d}y\,\mathrm{d}s_x$$

and the computation will performed when the integration cells are at a distance smaller than ε. We also note that no special implementation is needed since θ and v are evaluated on this own subdivisions.

Using the finite element method, establishing the *inf-sup* condition is essential to ensure the well-posedness of the discrete formulation. Letting H and h denote the mesh sizes of Γ and the background domain Ω, respectively, it is critical to demonstrate that the constant in the *inf-sup* condition remains independent of H and h. Previous work by Bramble [3] showed that the discrete *inf-sup* condition holds under the constraint $h \le CH$, where C is a sufficiently small constant. In two dimensions, Girault and Glowinski [11] demonstrated that $C = 1/3$ is valid for uniform background meshes. Dahmen and Kunoth [7] provided an abstract framework linking the constant in the *inf-sup* condition to the constants in the Trace theorem, norm equivalences for $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, space approximation estimates in Ω, and inverse inequalities for Γ.

In this paper, we investigate the well-posedness of the finite element approximation of the fictitious domain method when the coupling terms are approximated using the Dirac delta function. We demonstrate that under an additional condition, $\varepsilon \leq CH$ for some positive constant C, the discrete *inf-sup* condition holds.

The remainder of the paper is organized as follows: Section 2 introduces the variational formulation of the fictitious domain method and its finite element approximation. Essential analysis tools are also presented in this section. In Section 3, we review the discrete *inf-sup* condition and provide a stability analysis. Finally, Section 4 introduces a regularized version of the coupling term and establishes its discrete stability.

Notations

In what follows, we denote $H^r(\Omega)$ with $r \in [0,2]$ the standard Sobolev spaces. $H_0^1(\Omega)$ denotes the set of functions in $H^1(\Omega)$ with vanishing boundary. $H^{-r}(\Omega)$ is the dual of $H^r(\Omega) \cap H_0^1(\Omega)$, while $H^{-r}(\Gamma)$ denotes the dual of $H^r(\Gamma)$ for $r \in [0,1]$. All the duel spaces are equipped with the standard induced norms. We also set $H^{r^\pm}(\Omega) := H^{r\pm\varepsilon}(\Omega)$ for any small positive ε.

2 Preliminaries

Let Ω be a bounded background domain with Lipschitz boundary and ω is the physical domain inside Ω. We set $\Gamma = \partial\omega$. We also assume that Γ is Lipschitz and it is away from the boundary of Ω, namely, there exists a positive constant c_0 satisfying

$$\operatorname{dist}(\Gamma, \partial\Omega) > c_0. \tag{1}$$

Fictitious domain method

Let $\widetilde{f} \in L^2(\omega)$, and $g \in H^{1/2}(\Gamma)$, we consider the Poisson equation

$$\begin{aligned} -\Delta u &= \widetilde{f}, && \text{in } \omega, \\ u &= -g, && \text{on } \Gamma. \end{aligned}$$

Let $f \in L^2(\Omega)$ be an bounded extension of $\widetilde{f} \in L^2(\omega)$. Then, we shall consider the following variational formulation: given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, we want to find $(u,\lambda) \in H_0^1(\Omega) \times H^{-1/2}(\Gamma)$ satisfying

$$\begin{aligned} (\nabla u, \nabla v) + \langle \lambda, \mathtt{tr}\, v\rangle &= (f, v), && \text{for all } v \in H_0^1(\Omega), \\ \langle \theta, \mathtt{tr}\, u\rangle &= \langle \theta, g\rangle, && \text{for all } \theta \in H^{-1/2}(\Gamma). \end{aligned} \tag{2}$$

Here $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product; $\langle \cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$; and $\mathtt{tr}$ is the trace operator onto Γ. It is well known that

$\inf\{\|v\|_{H^1(\Omega)} : \mathtt{tr}\, v = w\}$ is an equivalent norm on $\|w\|_{H^{1/2}(\Gamma)}$ and hence there holds the *inf-sup* condition

$$\inf_{\theta\in H^{-1/2}(\Gamma)} \sup_{v\in H_0^1(\Omega)} \frac{\langle \theta, \mathtt{tr}\, v\rangle}{\|\theta\|_{H^{-1/2}(\Gamma)}\|v\|_{H^1(\Omega)}} \geq b.$$

The above *inf-sup* condition together with the coercivity of the Dirichlet form implies that problem (2) is well-posed (cf. [4]).

The trace operator

We shall use the following Trace Theorem for our analysis: given $v \in H^s(\omega)$ with $s \in (\frac{1}{2}, \frac{3}{2})$, we have

$$\|\,\mathtt{tr}\, v\|_{H^{s-1/2}(\Gamma)} \leq C_T\|v\|_{H^s(\omega)} \leq C_T\|v\|_{H^s(\Omega)}. \tag{3}$$

See e.g. [10] for a complete proof. It is also known that there exists an right inverse of the trace operator E such that $\mathtt{tr}\, E\theta = \theta$ for all $\theta \in H^{s-1/2}(\Gamma)$ and

$$\|E\theta\|_{H^s(\omega)} \leq C_{IT}\|\theta\|_{H^{s-1/2}(\Gamma)}. \tag{4}$$

We can continue to extend $E\theta$ to Ω using the Whitney extension E_W satisfying $E_W v|_\omega = v$ for all $v \in H^s(\omega)$ and $\|E_W v\|_\Omega \leq C_W\|v\|_{H^s(\Omega)}$. Setting the extension operator $\mathcal{E} = E_W E$, we obtain that for all $\theta \in H^{s-1/2}(\Gamma)$,

$$\|\mathcal{E}\theta\|_{H^s(\Omega)} \leq C_{ITW}\|\theta\|_{H^{s-1/2}(\Gamma)}, \tag{5}$$

where $C_{ITW} = C_{IT}C_W$.

Elliptic Regularity

Our estimates rely on standard regularity results for elliptic problems: given $g \in H^{-1}(\Omega)$, let $T : H^{-1}(\Omega) \to H_0^1(\Omega)$ be the solution operator satisfying

$$(\nabla Tg, \nabla v) = \langle g, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \qquad \text{for all } v \in H_0^1(\Omega). \tag{6}$$

We first note that if $g \in L^2(\Omega)$, we identify $\langle g, \cdot\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$ with $(g, \cdot)_\Omega$. The following assumption provides the regularity of Tg up to the boundary.

Assumption 1 (elliptic regularity)**.** *There exists* $r \in (\frac{1}{2}, 1]$ *and a positive constant* C_{reg} *satisfying*

$$\|Tg\|_{H^{1+r}(\Omega)} \leq C_{reg}\|g\|_{H^{-1+r}(\Omega)}.$$

As an example, consider the case where Ω is a polytope. Based the regularity results provided by [8], r in Assumption 1 is between $\frac{1}{2}$ and 1 and can be determined by the shape of Ω.

A discrete system

Now we assume that both Ω and Γ are polygonal. Let $\mathcal{T}_h(\Omega)$ and $\mathcal{T}_H(\Gamma)$ be quasi-uniform subdivisions of Ω and Γ, where h and H denotes the corresponding mesh sizes. Let $\mathbb{V}_h$ be the continuous piecewise linear finite element space subordinate to $\mathcal{T}_h(\Omega)$ and let $\mathbb{W}_H$ be the piecewise constant finite element space subordinate to $\mathcal{T}_H(\Gamma)$. The discrete system reads: find $(u_h, \lambda_H) \in \mathbb{V}_h \times \mathbb{W}_H$ such that

$$\begin{aligned} (\nabla u_h, \nabla v_h) + \langle \lambda_H, \mathtt{tr}\, v_h \rangle &= (f, v_h), \quad \text{for all } v_h \in \mathbb{V}_h, \\ \langle \theta_H, u_h \rangle &= \langle \theta_H, g \rangle, \quad \text{for all } \theta_H \in \mathbb{W}_H. \end{aligned} \tag{7}$$

Some estimates in discrete spaces

Define the elliptic projection $R_h : H_0^1(\Omega) \to \mathbb{V}_h$ with

$$(\nabla R_h v, \nabla w_h) = (\nabla v, \nabla w_h), \qquad \text{for all } w_h \in \mathbb{V}_h.$$

The following approximation property will be used in our stability analysis: given $v_h \in \mathbb{V}_h \subset H^{3/2^-}(\Omega)$, there holds that

$$\|(I - R_h) v_h\|_{H^1(\Omega)} \le C_{app} h^{1/2^-} \|v_h\|_{H^{3/2^-}(\Omega)}. \tag{8}$$

We shall also use the following inverse inequality for $\mathbb{W}_H$: given $\theta_H \in \mathbb{W}_H$, we have

$$\|\theta_H\|_{L^2(\Gamma)} \le C_{inv} H^{-1/2} \|\theta_H\|_{H^{-1/2}(\Gamma)}. \tag{9}$$

3 Discrete inf-sup condition for (7)

In order to prove the well-posedness of the discrete problem (7), we need to show the following *inf-sup* condition: there exists a positive constant β independent of h and H so that

$$\inf_{\theta_H \in \mathbb{W}_H} \sup_{v_h \in \mathbb{V}_h} \frac{\langle \theta_H, \mathtt{tr}\, v_h \rangle}{\|\theta_H\|_{H^{-1/2}(\Gamma)} \|v_h\|_{H^1(\Omega)}} \ge \beta. \tag{10}$$

In the following lemma, we provide a complete proof of (10) in the spirit of [3] (see also [7] a more general proof) and our proof for the regularized version will follow this argument. We also show how the above constant β depends on the constants mentioned in Section 2.

Lemma 2 (discrete *inf-sup* condition)**.** *Let H, the mesh size of $\mathcal{T}_H(\Gamma)$, be fixed. Assume that h, the mesh size of $\mathcal{T}_h(\Omega)$, small enough so that*

$$C_{app} C_{reg} C_{ITW} C_T C_{inv} h^{1/2^-} H^{-1/2} = \tfrac{1}{2}.$$

Then the inf-sup *condition* (10) *holds with* $\beta = \frac{1}{2C_{ITW}}$.

Proof. For a fixed $\theta_H \in \mathbb{W}_H$, we set $T\theta_H \in H_0^1(\Omega)$ to be the solution of the Laplace problem, i.e.

$$(\nabla T\theta_H, \nabla v) = \langle F_{\theta_H}, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} := \langle \theta_H, \mathtt{tr}\, v\rangle, \qquad \text{for all } v \in H_0^1(\Omega),$$

We note that $F_{\theta_H} \in H^{-1/2^-}(\Omega)$ in view of (3):

$$\begin{aligned} \|F_{\theta_H}\|_{H^{-1/2^-}(\Omega)} &= \sup_{v\in H^{1/2^+}(\Omega)} \frac{\langle \theta_H, \mathtt{tr}\, v\rangle}{\|v\|_{H^{1/2^+}(\Omega)}} \\ &\leq \frac{\|\theta_H\|_{L^2(\Gamma)}\|\,\mathtt{tr}\, v\|_{L^2(\Gamma)}}{\|v\|_{H^{1/2^+}(\Omega)}} \leq C_T\|\theta_H\|_{L^2(\Gamma)}. \end{aligned} \tag{11}$$

Hence by elliptic regularity, $T\theta_H \in H^{3/2^-}(\Omega)$. Let $v = R_hT\theta_H$ to get

$$\begin{aligned} \langle \theta_H, \mathtt{tr}\, R_hT\theta_H\rangle &= (\nabla T\theta_H, \nabla R_hT\theta_H) \\ &= (\nabla R_hT\theta_H, \nabla R_hT\theta_H) = \|R_hT\theta_H\|^2_{H^1(\Omega)}. \end{aligned}$$

On the other hand, thanks to (5), there holds that

$$\begin{aligned} \|\theta_H\|_{H^{-1/2}(\Gamma)} &= \sup_{w\in H^{1/2}(\Gamma)} \frac{\langle \theta_H, w\rangle}{\|w\|_{H^{1/2}(\Gamma)}} \\ &= \sup_{w\in H^{1/2}(\Gamma)} \frac{(\nabla T\theta_H, \nabla \mathcal{E}w)}{\|w\|_{H^{1/2}(\Gamma)}} \leq \sup_{w\in H^{1/2}(\Omega)} \frac{\|T\theta_H\|_{H^1(\Omega)}\|\mathcal{E}w\|_{H^1(\Omega)}}{\|w\|_{H^{1/2}(\Gamma)}} \\ &\leq C_{ITW}\|T\theta_H\|_{H^1(\Omega)}. \end{aligned} \tag{12}$$

Next we want to bound $T\theta_H$. The approximation property of the elliptic projection (8) together with Assumption 1 and (11) implies that

$$\begin{aligned} \|T\theta_H\|_{H^1(\Omega)} &\leq \|(I - R_h)T\theta_H\|_{H^1(\Omega)} + \|R_hT\theta_H\|_{H^1(\Omega)} \\ &\leq C_{app}h^{1/2^-}\|T\theta_H\|_{H^{3/2^-}(\Omega)} + \|R_hT\theta_H\|_{H^1(\Omega)} \\ &\leq C_{app}C_{reg}h^{1/2^-}\|F_{\theta_H}\|_{H^{-1/2^-}(\Omega)} + \|R_hT\theta_H\|_{H^1(\Omega)} \\ &\leq C_{app}C_{reg}C_Th^{1/2^-}\|\theta_H\|_{L^2(\Gamma)} + \|R_hT\theta_H\|_{H^1(\Omega)} \\ &\leq C_{app}C_{reg}C_TC_{inv}h^{1/2^-}H^{-1/2}\|\theta_H\|_{H^{-1/2}(\Gamma)} + \|R_hT\theta_H\|_{H^1(\Omega)}, \end{aligned}$$

where for the last inequality we used the inverse equality (9) for $\mathbb{W}_H$. Combing the above estimate and (12) to obtain that

$$\begin{aligned} \|\theta_H\|_{H^{-1/2}(\Gamma)} \leq\ & C_{app}C_{reg}C_{ITW}C_TC_{inv}h^{1/2^-}H^{-1/2}\|\theta_H\|_{H^{-1/2}(\Gamma)} \\ &+ C_{ITW}\|R_hT\theta_H\|_{H^1(\Omega)}. \end{aligned}$$

Now we let h small enough so that $C_{app}C_{reg}C_{ITW}C_TC_{inv}h^{1/2^-}H^{-1/2} = \frac{1}{2}$. This implies that

$$\|\theta_H\|_{H^{-1/2}(\Gamma)} \leq 2C_{ITW}\|R_hT\theta_H\|_{H^1(\Omega)}.$$

Combing all the results, we conclude that for all $\theta_H \in \mathbb{W}_H$,

$$\langle \theta_H, \mathtt{tr}\, R_hT\theta_H\rangle \geq \frac{1}{2C_{ITW}}\|R_hT\theta_H\|_{H^1(\Omega)}\|\theta_H\|_{H^{-1/2}(\Gamma)}.$$

The proof is complete. □

4 A variational formulation with regularization

In this section, we consider a new variational formulation by regularizing the function defined on Γ using a class of approximations of the Dirac distributions.

Regularization

Let ψ_{1d} be a function in $C^1(\mathbb{R})$ such that ϕ_{1d} is compactly supported in $(-1,1)$. Let $\psi(x) = c\psi(|x|)$ for some constant c so that $\int_{\mathbb{R}^d} \psi \,\mathrm{d}x = 1$. Then we set

$$\delta^\varepsilon(x) = \frac{1}{\varepsilon^d}\psi\left(\frac{x}{\varepsilon}\right). \tag{13}$$

For a function $v \in L^1(\Omega)$, we define its regularization by

$$v^\varepsilon(x) := \int_\Omega \delta^\varepsilon(x-y)v(y)\,\mathrm{d}y, \qquad \text{for } x \in \Omega. \tag{14}$$

For a functional $F \in H^{s-1}(\omega) \cap H^{-1}(\Omega)$ with $s \in [0,1]$. We define its regularization F^ε satisfying

$$\langle F^\varepsilon, v\rangle_{H^{-1}(\Omega),H^1(\Omega)} = \langle F, v^\varepsilon\rangle_{H^{-1}(\Omega),H^1(\Omega)}, \qquad \text{for all } v \in H^1(\Omega). \tag{15}$$

In particular, given $\theta_H \in \mathbb{W}_H$, we can define the functional F_{θ_H} by

$$F^\varepsilon_{\theta_H} = \int_\Gamma \theta_H(y)\delta^\varepsilon(x-y)\,\mathrm{d}y =: \theta^\varepsilon_H,$$

noting that here we used the fact that δ^ε is radially symmetric.

Remark 3. *We note that one can also generate the Dirac delta approximation by the tensor product of ψ_{1d} and choose ψ_{1d} suitably with certain moment condition and smoothness condition. For more choices we refer to [12, 13].*

A regularized discrete formulation

The regularized discrete formulation reads: find $(U_h, \Lambda_H) \in \mathbb{V}_h \times \mathbb{W}_H$ satisfying

$$\begin{aligned} (\nabla U_h, \nabla v_h) + (\Lambda^\varepsilon_H, v_h) &= (f, v_h), \quad \text{for all } v_h \in \mathbb{V}_h, \\ (\theta^\varepsilon_H, U_h) &= \langle \theta_H, g\rangle, \quad \text{for all } \theta_H \in \mathbb{W}_H. \end{aligned} \tag{16}$$

Stability

Following the argument of Lemma 2, we are ready to show the *inf-sup* condition for the above discrete system. Next we show that the above discrete system is well-posed.

Theorem 4. *(discrete inf-sup condition for* (16)*) Let H be fixed. There exist two positive constants C_1 and C_2 such that when $\varepsilon \le C_1 H$ and $h \le C_2 H$, there holds*

$$\inf_{\theta_H \in \mathbb{W}_H} \sup_{v_h \in \mathbb{V}_h} \frac{(\theta_H^\varepsilon, v_h)}{\|\theta_H\|_{H^{-1/2}(\Gamma)} \|v_h\|_{H^1(\Omega)}} \ge \widetilde{\beta}, \tag{17}$$

where $\widetilde{\beta}$ is a positive constant independent of ε, h and H.

Proof. We first try to bound θ_H in $H^{1/2^-}(\Gamma)$ norm by $T\theta_H^\varepsilon$. According to Theorem 3 of [12] there holds

$$\|F_{\theta_H} - F_{\theta_H}^\varepsilon\|_{H^{-1}(\Omega)} \le C_r \varepsilon^{1/2} \|F_{\theta_H}\|_{H^{-1/2^-}(\omega)}.$$

Following (11), we can also show that $\|F_{\theta_H}\|_{H^{-1/2^-}(\omega)} \le C_T \|\theta_H\|_{L^2(\Gamma)}$. Thus,

$$\begin{aligned}
\|T\theta_H\|_{H^1(\Omega)} &\le \|T\theta_H - T\theta_H^\varepsilon\|_{H^1(\Omega)} + \|T\theta_H^\varepsilon\|_{H^1(\Omega)} \\
&\le \|F_{\theta_H} - F_{\theta_H}^\varepsilon\|_{H^{-1}(\Omega)} + \|T\theta_H^\varepsilon\|_{H^1(\Omega)} \\
&\le C_r C_T \varepsilon^{1/2} \|\theta_H\|_{L^2(\Gamma)} + \|T\theta_H^\varepsilon\|_{H^1(\Omega)} \\
&\le C_r C_T C_{inv} \varepsilon^{1/2} H^{-1/2} \|\theta_H\|_{H^{-1/2}(\Gamma)} + \|T\theta_H^\varepsilon\|_{H^1(\Omega)}.
\end{aligned}$$

Combing the above estimate with (12) in Lemma 2 and set

$$C_r C_{ITW} C_T C_{inv} \varepsilon^{1/2} H^{-1/2} = \tfrac{1}{2}$$

to obtain that

$$\|\theta_H\|_{H^{-1/2}(\Gamma)} \le 2 C_{ITW} \|T\theta_H^\varepsilon\|_{H^1(\Omega)}. \tag{18}$$

On the other hand, according to the definition of the regulation of the a linear functional acting on $H^{-1/2}(\Omega)$ and the trace inequality, there holds that

$$\begin{aligned}
\|\theta_H^\varepsilon\|_{H^{-1/2-}(\Omega)} &= \sup_{w \in H^{1/2^+}(\Omega)} \frac{(\theta_H^\varepsilon, w)}{\|w\|_{H^{1/2^+}(\Omega)}} = \sup_{w \in H^{1/2^+}(\Omega)} \frac{\langle \theta_H, w^\varepsilon \rangle}{\|w\|_{H^{1/2^+}(\Omega)}} \\
&\le \sup_{w \in H^{1/2^+}(\Omega)} \frac{\|\theta_H\|_{L^2(\Gamma)} \|w^\varepsilon\|_{L^2(\Gamma)}}{\|w\|_{H^{1/2^+}(\Omega)}} \\
&\le C_T \sup_{w \in H^{1/2^+}(\Omega)} \frac{\|\theta_H\|_{L^2(\Gamma)} \|w^\varepsilon\|_{H^{1/2^+}(\omega)}}{\|w\|_{H^{1/2^+}(\Omega)}} \le C_T C_{rs} \|\theta_H\|_{L^2(\Gamma)}.
\end{aligned}$$

Here for the last inequality above, we again used Theorem 3 of [12]:

$$\|w - w^\varepsilon\|_{H^{1/2^+}(\omega)} \le C_r \|w\|_{H^{1/2^+}(\Omega)}$$

so that

$$\|w^\varepsilon\|_{H^{1/2^+}(\omega)} \le C_{rs} \|w\|_{H^{1/2^+}(\Omega)}$$

with $C_{rs} = 1 + C_r$.

Utilizing the above estimate together with (18), we continue to bound $T\theta_H^\varepsilon$ by

$$\begin{aligned}
\|T\theta_H^\varepsilon\|_{H^1(\Omega)} &\le \|(I-R_h)\theta_H^\varepsilon\|_{H^1(\Omega)} + \|R_h T\theta_H^\varepsilon\|_{H^1(\Omega)} \\
&\le C_{app} h^{1/2^-} \|T\theta_H^\varepsilon\|_{H^{3/2^-}(\Omega))} + \|R_h T\theta_H^\varepsilon\|_{H^1(\Omega)} \\
&\le C_{app} C_{reg} h^{1/2^-} \|\theta_H^\varepsilon\|_{H^{-1/2^-}(\Omega)} + \|R_h T\theta_H^\varepsilon\|_{H^1(\Omega)} \\
&\le C_{app} C_{reg} C_T C_{rs} h^{1/2^-} \|\theta_H\|_{L^2(\Gamma)} + \|R_h T\theta_H^\varepsilon\|_{H^1(\Omega)} \\
&\le C_{app} C_{reg} C_T C_{rs} C_{inv} h^{1/2^-} H^{-1/2} \|\theta_H\|_{H^{-1/2}(\Gamma)} + \|R_h T\theta_H^\varepsilon\|_{H^1(\Omega)}.
\end{aligned}$$

Now we set

$$2C_{app} C_{reg} C_T C_{rs} C_{ITW} C_{inv} h^{1/2^-} H^{-1/2} = \tfrac{1}{2}$$

to get

$$\|\theta_H\|_{H^{-1/2}(\Gamma)} \le 4C_{ITW} \|R_h T\theta_H^\varepsilon\|_{H^1(\Omega)}.$$

We end this proof by setting $v_h = R_h T\theta_H^\varepsilon$ in (17) so that

$$\frac{(\theta_H^\varepsilon, v_h)}{\|\theta_H\|_{H^{-1/2}(\Gamma)} \|v_h\|_{H^1(\Omega)}} = \frac{\|R_h T\theta_H^\varepsilon\|^2_{H^1(\Omega)}}{\|\theta_H\|_{H^{-1/2}(\Gamma)} \|R_h T\theta_H^\varepsilon\|_{H^1(\Omega)}} \ge \frac{1}{4C_T}.$$

□

References

[1] Ivo Babuška. The finite element method with penalty. *Mathematics of computation*, 27(122):221–228, 1973.

[2] Christoph Börgers and Olof B Widlund. On finite element domain imbedding methods. *SIAM journal on numerical analysis*, 27(4):963–978, 1990.

[3] James H. Bramble. The Lagrange multiplier method for Dirichlet's problem. *Math. Comp.*, 37(155):1–11, 1981.

[4] Franco Brezzi and Michel Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[5] Erik Burman and Peter Hansbo. Fictitious domain finite element methods using cut elements: I. a stabilized lagrange multiplier method. *Computer Methods in Applied Mechanics and Engineering*, 199(41-44):2680–2686, 2010.

[6] Erik Burman and Peter Hansbo. Fictitious domain finite element methods using cut elements: II. a stabilized nitsche method. *Applied Numerical Mathematics*, 62(4):328–341, 2012.

[7] Wolfgang Dahmen and Angela Kunoth. Appending boundary conditions by Lagrange multipliers: analysis of the LBB condition. *Numer. Math.*, 88(1):9–42, 2001.

[8] Monique Dauge. *Elliptic boundary value problems on corner domains*, volume 1341 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988. Smoothness and asymptotics of solutions.

[9] Françoise Demengel and Gilbert Demengel. *Functional spaces for the theory of elliptic partial differential equations*. Universitext. Springer, London; EDP Sciences, Les Ulis, 2012. Translated from the 2007 French original by Reinie Erné.

[10] Zhonghai Ding. A proof of the trace theorem of Sobolev spaces on Lipschitz domains. *Proc. Amer. Math. Soc.*, 124(2):591–600, 1996.

[11] Vivette Girault and Roland Glowinski. Error analysis of a fictitious domain method applied to a dirichlet problem. *Japan Journal of Industrial and Applied Mathematics*, 12:487–514, 1995.

[12] Luca Heltai and Wenyu Lei. A priori error estimates of regularized elliptic problems. *Numerische Mathematik*, 146(3):571–596, 2020.

[13] Luca Heltai and Wenyu Lei. Adaptive finite element approximations for elliptic problems using regularized forcing data. *SIAM Journal on Numerical Analysis*, 61(2): 431–456, 2023.

[14] Charles S Peskin. The immersed boundary method. *Acta numerica*, 11:479–517, 2002.