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Modeling Aeroelastic Phenomena via Stochastic Resonance in Nonlinear Bistable Oscillators

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Abstract

Stochastic resonance is a phenomenon where noise enhances a nonlinear system's ability to respond to weak periodic excitation. This effect is particularly relevant in bistable systems encountered in post-critical aeroelastic conditions, where purely deterministic models fail to capture observed transitions under unsteady aerodynamic loading. This study explores stochastic resonance in a nonlinear Duffing-type oscillator driven by harmonic forcing and additive white noise, representing a reduced model of a prismatic beam in crossflow, inspired by wind tunnel tests on bridge-like structures. The paper complements a previously developed approximative framework for analysis of the Fokker–Planck equation, which employs a periodic expansion linked with the method of stochastic moments. This approach provides a detailed and structured view of the evolving probability density, offering greater interpretability than standard black-box finite element methods, which are also used for comparison. The results are examined in detail against a FEM benchmark, and potential directions for improving the method—particularly with respect to transient accuracy and numerical stability—are outlined.

Keywords: stochastic resonance, aeroelasticity, Fokker-Planck equation, Duffing oscillator, numerical simulation, semi-analytical methods

1 Introduction

Stochastic resonance (SR) is a nonlinear phenomenon wherein random noise interacts constructively with a weak periodic signal to amplify a system's response. Originally introduced in the context of Brownian motion by Kramers [1] and later applied to climate models by Nicolis [2, 3], SR has since been widely explored across various scientific domains, as reviewed in [4, 5]. It typically manifests in bistable systems, where noise-driven transitions between quasi-stable states become synchronized with an external periodic input.

The relevance of SR spans disciplines such as optics, plasma physics, chemistry, biophysics, and signal processing. In engineered systems, SR can be seen both as a beneficial mechanism—for example in enhancing sensor sensitivity—or as a destabilizing factor, especially in contexts like aeroelasticity and plasma confinement, where it may contribute to unwanted vibrations or loss of stability. For practical applications and review in the engineering context, see [6].

Within aeroelastic research, SR has been proposed as a plausible explanation for certain post-critical dynamic behaviors. Although not yet a widely confirmed phenomenon in this field, the concept offers a compelling framework for interpreting some observed transitions in slender structures subjected to wind loading. Wind tunnel experiments on two-dimensional rectangular-section models have documented complex dynamics, such as divergence, buffeting, and mode switching, that are consistent with SR-like effects. These behaviors can be modeled by bistable nonlinear oscillators subjected to combined harmonic and stochastic forcing, as discussed in [7].

Motivated by this hypothesis, the present study investigates SR as a modeling tool for post-critical aeroelastic effects in prismatic beams exposed to crossflow. The system is represented as a single-degree-of-freedom (SDOF) Duffing oscillator with additive white noise and harmonic excitation, capturing the essence of noise-induced switching and resonance amplification in a simplified yet physically meaningful manner.

This work extends the earlier research presented in [8], where the authors analyzed the Fokker–Planck equation (FPE) associated with a Duffing-type oscillator under harmonic and stochastic forcing. That study combined semi-analytical and numerical approaches—including Galerkin-based FPE solutions and stochastic simulations—to investigate both stationary and transient dynamics. The present work builds on these foundations by analyzing the Galerkin-based solution in greater detail, aiming to assess its practical applicability and limitations more thoroughly than was possible to include in the original publication. It appears that the numerical techniques used exhibit notable drawbacks. In particular, the application of orthogonal Hermite polynomials as basis functions in the Galerkin framework occasionally leads to non-physical results, such as negative probability densities.

To address these shortcomings, exponential polynomial basis functions have been proposed as a promising alternative [9]. These functions offer improved approximation capabilities, particularly in capturing the tails of probability distributions. Recent developments in extending this approach to nonlinear, non-stationary systems [10], along with refinements that outperform Monte Carlo simulations in terms of accuracy [11], further underscore its potential. Another possible direction is the use of hierarchical basis functions such as orthogonal splines. Nonetheless, implementing and testing these methodologies lies beyond the scope of the present paper.

The stochastic moment method—essentially a Galerkin scheme with globally supported basis functions—retains several advantages despite its numerical challenges. While it demands the integration of high-order polynomials, potentially affecting stability and efficiency, it also facilitates access to multiple levels of approximation (moments) that enrich the interpretability of the resulting solution. This layered structure makes it a compelling alternative to black-box finite element method (FEM) solvers or highly specialized methods like those in [12]. In this study, the proposed method is directly compared to a reference FEM solution computed using the standard `NDSolve` function in Wolfram Mathematica, serving as a benchmark for both transient and cyclo-stationary behavior.

The structure of the paper is as follows. After this introduction, the governing mathematical model is presented. This is followed by a description of the semi-analytical method based on [8], with selected derivation steps omitted for brevity. Subsequently, the results for a specific case study are compared in detail with the FEM solution and critically discussed. Finally, conclusions are drawn and suggestions for future improvements are outlined.

2 Theoretical Model

To capture the essential features of the aeroelastic systems under consideration, the full two-dimensional aeroelastic problem is progressively reduced to a manageable, yet representative, one-dimensional nonlinear dynamical model.

A nonlinear bistable oscillator of the Duffing type is adopted, governed by the following differential equation:

$$\ddot{u} + 2\omega_b \dot{u} + V'(u) = P(t) + \xi(t) \,. \tag{1}$$

where $V(u)$ is the potential energy introduced in a form corresponding with the Duffing equation:

$$V(u) = -\frac{\omega_0^2}{2}u^2 + \frac{\gamma^4}{4}u^4 \qquad \Rightarrow \qquad V'(u) = dV(u)/du = -\omega_0^2 \cdot u + \gamma^4 \cdot u^3 \,, \tag{2}$$

and $\xi(t)$ is the Gaussian white noise of intensity $2\sigma^2$ respecting conditions:

$$\mathbf{E}\{\xi(t)\} = 0 \;; \qquad \mathbf{E}\{\xi(t)\xi(t')\} = 2\sigma^2 \cdot \delta(t - t') \,. \tag{3}$$

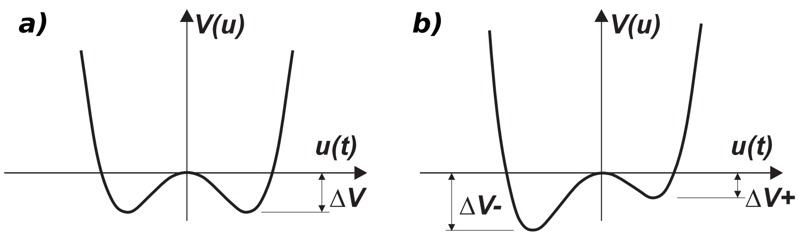

Figure 1: Bistable nonlinear system: a) Symmetric – b) Non-symmetric potential.

On the right hand side, the external excitation is defined as a harmonic force per unit mass, with P_0 denoting the amplitude and Ω the excitation frequency,

$$P(t) = P_0 \exp \mathrm{i}\Omega t\,. \tag{4}$$

This system exhibits bistability, characterized by two potential wells separated by an unstable equilibrium. When the amplitude and frequency of external harmonic excitation and the noise intensity are optimally tuned, SR can emerge. In SR, noise facilitates synchronized transitions between wells in response to periodic forcing.

The system's linear dynamics are governed by typical circular frequencies, ω_0 (eigenfrequency) and ω_b (damping). The linear component of the potential derivative $V'(u)$ renders the origin metastable, while a cubic term stabilizes the system at larger displacements. Figure 1 illustrates two cases: (a) a symmetric potential with equal energy barriers for both directions, (b) an asymmetric potential, influenced by an added linear stiffness, possibly shifting the system towards monostability.

The response described by Eq. (1) can be significantly amplified by introducing an optimal noise level, yielding SR. This concept, rooted in nonlinear optics, is analogous to photon-assisted inter-well hopping [4]. For a symmetric bistable potential, and in the absence of periodic forcing, the escape is approximated by the Kramers rate [1]:

$$\omega_e = \sqrt{2}\,\exp(-\Delta V/T)\,. \tag{5}$$

Here, ΔV is the barrier height and T the absolute temperature, which is proportional to the noise intensity σ^2 in Eq. (3). Though developed for nonlinear optical systems, the Kramers formula has since found broad application across different fields, [13], in both theoretical and experimental contexts.

2.1 Fokker–Planck Equation Analysis

The semi-analytical method evaluates the response probability density function (PDF) of system (1), assuming that both the input and output behave as Markov processes. This assumption permits the application of the FPE as the primary analytical tool. A semi-analytical solution is pursued using the stochastic moment method with non-Gaussian closure, under the assumption of a multiharmonic, periodic response.

Since the noise in Eq. (1) is additive, no Wong-Zakai correction is required [14,15]. The corresponding FPE can therefore be directly formulated, as outlined in [16]. For the SDOF system, the FPE for the PDF $p = (u, v, t)$, where $v = \dot{u}$, is given by:

$$\frac{\partial p}{\partial t} = -v\frac{\partial p}{\partial u} + \frac{\partial}{\partial v}\left(\left(2\omega_b \cdot v + V'(u) - P(t)\right)p\right) + \sigma^2\frac{\partial^2 p}{\partial v^2} \tag{6}$$

together with boundary and initial conditions:

$$\lim_{u,v\to\pm\infty} p(u, v, t) = 0\,, \tag{7a}$$

$$p(u, v, 0) = \delta(u, v)\,. \tag{7b}$$

Assuming $P(t) = 0$ and that the excitation is purely stationary random, the right-hand side of Eq. (6) becomes time-independent. In this case, the Fokker-Planck equation reduces to its stationary form and admits a Boltzmann-type solution [17, 18]:

$$p_0(u, v) = D\exp\left(-\frac{2\omega_b}{\sigma^2}H(u, v)\right)\,. \tag{8}$$

Here, D is a normalization constant, and $H(u, v)$ is the system's Hamiltonian:

$$H(u, v) = \frac{1}{2}v^2 + V(u) = \frac{1}{2}v^2 - \frac{1}{2}\omega_0^2u^2 + \frac{1}{4}\gamma^4u^4\,. \tag{9}$$

This leads to a separable solution:

$$p_0(u, v) = p_u(u)\,p_v(v)\,, \tag{10}$$

$$p_u(u) = D_u\exp\left(-\frac{2\omega_b}{\sigma^2}\left(\frac{1}{2}\omega_0^2u^2 + \frac{1}{4}\gamma^4u^4\right)\right)\,, \tag{11}$$

$$p_v(v) = D_v\exp\left(-\frac{2\omega_b}{\sigma^2}\frac{1}{2}v^2\right)\,. \tag{12}$$

This shows that u and v are stochastically independent in the stationary regime.

For both random and periodic periodic loading, the stationary, time-independent Boltzmann-type expression from Eq. (8) can be used as a base, modulated by a space- and time-dependent series. For large times ($t \to \infty$), the PDF may become periodic or cyclo-stationary in time.

Given the linearity of the Fokker-Planck equation, the periodicity of the PDF is expected to align with the frequency Ω of the deterministic excitation and its harmonics. Thus, the solution may be approximated by the following Fourier-type expansion:

$$p(u, v, t) = p_0(u, v)\sum_{j=0}^{J} q_j(u, v)\,\exp(\mathrm{i}\,j\Omega\,t)\,. \tag{13}$$

Here, $p_0(u, v)$ is the stationary component from Eq. (8), and Ω denotes the frequency of the harmonic excitation. This series constitutes a weak solution to the FPE that is periodic with period $T = 2\pi/\Omega$, providing a valid representation within a single cycle. To incorporate transient behavior or non-stationary dynamics in the PDF, an alternative expansion is required—one based on modified coefficient functions $q_j(u, v, t)$ that retain explicit time dependence rather than assuming a fixed periodic structure. For such purposes, the standard approach based on the method of lines may be employed.

The unknown functions $q_j(u, v)$ in Eq. (13) can be determined using the generalized method of stochastic moments, as described in [16]. Applying the Galerkin method, Eq. (13) is substituted into the Fokker-Planck equation (6), and the resulting expression is projected onto a set of testing functions $\alpha(u, v)$ by multiplying both sides accordingly.

In [8], the functions $\alpha(u, v)$ and the expansions for $q_j(u, v)$ are chosen in the following polynomial-Hermite form, which offers analytical advantages:

$$\alpha(u,v) = \alpha_{r,s}(u,v) = u^r\, H_s(\beta v)\,, \qquad r = 0,\dots,R; \quad s = 0,\dots,S\,, \tag{14}$$

$$q_j(u,v) = \sum_{k=0}^{R}\sum_{l=0}^{S} q_{j,kl}\, u^k\, H_l(\beta v)\,, \qquad \beta = \sqrt{\frac{\omega_b}{\sigma^2}}\,, \tag{15}$$

where $H_s(\beta v)$ denotes the (physicists') Hermite polynomials, and the parameter β ensures proper scaling with respect to the noise intensity; R and S relate to the total polynomial degree.

The coefficients $q_{j,kl}$ are determined by taking the mathematical expectation with respect to the base PDF $p_o(u, v)$ from Eq. (8). The solution procedure leverages the orthogonality of the Hermite polynomials and applies a three-term recurrence relation, as detailed in [8].

2.2 Performance of the Approximative Periodic Solution

The performance of the approximative periodic solution, derived via the Galerkin stochastic moment method, was evaluated against a reference FEM solution computed using standard `NDSolve` command in the Wolfram Mathematica 12.3. All solutions were computed for the system defined by the parameters: $\omega_0^2 = 1$, $\gamma^4 = 0.5$, $\omega_b = 0.25$, $P_0 = 0.38$, and $\Omega = 0.09$, corresponding to a resonant noise intensity of $\sigma = 0.36$. The maximal polynomial degree in the expansions was limited to $R\,S = 8$.

The Fokker-Planck equation (6) was solved over a spatial-temporal domain large enough to include all regions with non-negligible probability. The initial condition was taken as the stationary Boltzmann-type distribution described in Eq. (8).

Figure 2 provides a detailed comparison between the FEM solution and the stochastic moment method with two different resolutions. Each row in the figure corresponds to a different computational approach: the first row *(a, b)* presents the FEM benchmark solution; the second row *(c, d)* shows the PDF computed using the stochastic moment method with $J = 5$ frequency term; and the third row *(e, f)* corresponds to the stochastic moment method with $J = 10$ terms. In each row, the left column *(a, c, e)* displays a 3D surface plot of the marginal PDF of displacement u as it evolves over

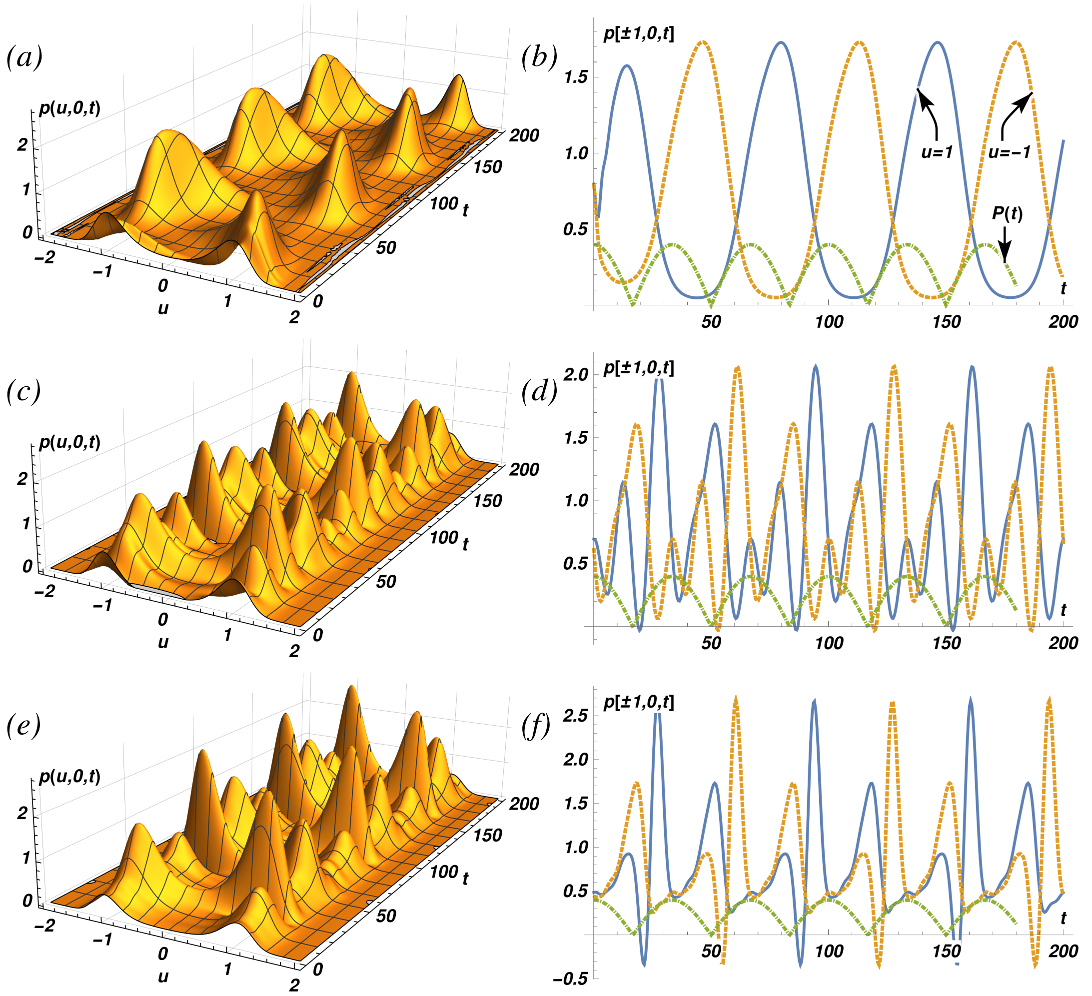

Figure 2: Comparison of marginal PDF evolution of u obtained via FEM (top row), the stochastic moment method with $J = 5$ (middle row), and $J = 10$ (bottom row). Left column: 3D plots of PDF evolution over time. Right column: PDF cross-sections at $u = \pm 1$ (solid blue and dashed yellow), with periodic forcing overlaid (green, dot-dashed).

time. The right column *(b, d, f)* shows temporal cross-sections of the marginal PDF at $u = \pm 1$, with the values at $u = 1$ and $u = -1$ shown as solid blue and dashed yellow lines, respectively. For reference, the external periodic forcing is overlaid as a green dot-dashed line.

The stochastic moment method shows some agreement with the FEM solution in capturing the timing and localization of the probability maxima, providing qualitatively similar behavior. Nevertheless, some notable discrepancies emerge. The Galerkin approximation can produce negative probability values, especially at higher harmonic resolutions, which are non-physical artifacts arising from polynomial overfitting. Additionally, the inclusion of more harmonics tends to induce unrealistically sharp features in the PDF, compromising smoothness and physical plausibility. While reducing the number of basis elements from $J = 10$ to 5 mitigates these negative values, it also introduces visible oscillations in the PDF, indicating aliasing due to under-resolution. Increasing J beyond 10 yields no clear improvements, suggesting that the method reaches a saturation point in accuracy with respect to modal resolution.

Furthermore, the moment method solutions exhibit a clear phase shift with respect to the FEM benchmark, despite both being initialized from the same initial condition. While this shift may not critically affect the analysis of cyclo-stationary properties of the established periodic solution, it highlights a limitation of the method when capturing transient dynamics. For applications where accurate modeling of time-dependent behavior is essential, especially during the approach to steady state, a more advanced or time-accurate numerical method would be necessary.

3 Conclusion

This study examined the response of a non-linear single-degree-of-freedom system with cubic stiffness characteristics under the influence of combined random and periodic excitation. A semi-analytical method, previously developed to approximate the effects of stochastic resonance—based on the generalized method of stochastic moments—was here subjected to a more detailed theoretical and numerical evaluation.

The corresponding Fokker-Planck equation for the Duffing-type oscillator was solved approximately, yielding a multiharmonic, periodic probability density function that captured both fundamental and higher-order harmonics of the system's non-stationary behavior. While the method successfully revealed key qualitative features of the response, including resonance amplification, its quantitative reliability was limited by numerical artifacts such as non-physical negative probabilities introduced by polynomial interpolation.

To assess accuracy, a reference solution was computed using a time-dependent finite element method within a general-purpose numerical solver. This numerical benchmark confirmed the structure and existence of the stochastic resonance regime, highlighting both the strengths and the limitations of the semi-analytical model.

Several directions for refinement were identified. Replacing the polynomial interpolation with alternative function bases, such as exponential-polynomial expansions or orthogonal splines, may improve numerical stability, convergence, and the physical fidelity of the PDF, especially in regions with sharp gradients or localized features.

Although this work focused on cubic nonlinearities and additive noise, the framework can be extended to more complex systems, including asymmetric or non-polynomial nonlinearities. Future studies could also explore alternative analytical tools, such as the Floquet theorem or the maximum entropy principle, to enhance the modeling of noise-driven phenomena.

Finally, these findings hold potential for broader applications, particularly in fields like post-critical aeroelasticity and vehicle dynamics, where understanding and controlling stochastic resonance effects could provide new strategies for mitigating or harnessing noise-induced response amplification in engineering systems.

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