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Shape Optimization of Reticulated Shells with Loading Uncertainty via a Monte Carlo Approach

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Abstract

In this contribution the design of reticulated shells is dealt with, exploring the optimal solutions that can be retrieved by a form-finding approach in the case of loading uncertainty. To this goal, a numerical tool is implemented that addresses the design of reticulated shells through funicular analysis. The force density method (FDM) is implemented to cope with the equilibrium of reticulated shells whose branches are required to behave as bars. Optimal networks are sought by coupling FDM with techniques of sequential convex programming that were originally conceived to handle formulations of size optimization for elastic structures. The mean compliance computed across a set of statistical samples is used as objective function to be minimized, whereas constraints on the total length of the branches and the standard deviation of the length of its members are enforced. Funicular networks that are fully feasible with respect to the set of enforcements are retrieved. Optimal solutions are explored for a dome, considering uncertainties affecting self-weight.

Keywords: form-finding, structural optimization, force density method, uncertain loading, lattice domes, Monte Carlo simulations

1 Introduction and motivation

Reticulated shells have double curvature, while consisting of branches that mainly undergo axial forces [1, 2].

Optimal shapes for this kind of structures can be sought, among the others, by means of equilibrium-based methods, e.g. funicular analysis, see in particular [3, 4, 5]. Lattice shells are addressed as statically indeterminate networks of vertices and edges with given connectivity. Supports are provided at the restrained nodes, whereas unrestrained ones are in equilibrium with the applied nodal loads. Exploiting the concept of force densities, i.e. the ratio of force to length in each branch of the network [6], the equilibrium equations for the loaded nodes are uncoupled in the three spatial directions and linear in the control parameters.

In this contribution, a generalization of the approach presented in [7] is adopted, by embedding the Force Density Method (FDM) within a multi-constrained minimization problem. Due to its peculiar form, this problem can be efficiently solved through techniques of sequential convex programming [8] that were conceived to handle multi-constrained formulations of size optimization for elastic structures, see [9] among the others.

The research of the optimal shape of reticulated shells is made by considering loading uncertainty relying on a Monte Carlo approach. Having the main aim of preliminary investigating uncertainties affecting self-weight, a set of statical samples is generated considering normal uncorrelated point loads. The average value is the deterministic weight of the bars falling in the tributary area, whereas several coefficients of variations are assigned. An elastic solution is performed for each one of the generated load cases, see in particular [10], and the average value of the compliance is adopted as objective function. Constraints are of geometric type, being related to the minimum and maximum value of the total length of the network and to the standard deviation of the length of its members.

In the next sections, a brief overview of the force density method (FDM) is given, and the multi-constrained problem is presented. A numerical example is shown to demonstrate the method and draw some preliminary conclusions.

2 Force density method

FDM, i.e. the "force density method" [6], is used to handle the equilibrium of spatial networks.

A funicular network is made of $n_s = n + n_f$ nodes and m branches, which can be struts either ties. The axes of the Cartesian reference system with origin O are denoted by x, y, and z. The vectors $\mathbf{x}_s$, $\mathbf{y}_s$, $\mathbf{z}_s$ gather the coordinates of the n_s nodes: $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ refer to the n unrestrained nodes, i.e. the nodes subject to external forces; $\mathbf{x}_f$, $\mathbf{y}_f$, $\mathbf{z}_f$ collect the n_f restrained nodes, i.e. those where reactions arise. The connectivity matrix that provides the shape of the grid is $\mathbf{C}_s$, having subset $\mathbf{C}$ for the unrestrained nodes and $\mathbf{C}_f$ for the restrained ones. The vectors gathering the coordinate difference of the nodes along the axis x, y, z are denoted by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, respectively:

$$\mathbf{u} = \mathbf{C}_s\mathbf{x}_s, \quad \mathbf{v} = \mathbf{C}_s\mathbf{y}_s, \quad \mathbf{w} = \mathbf{C}_s\mathbf{z}_s. \tag{1}$$

The force densities, i.e. the ratios force to length for each branch of the network, are collected in $\mathbf{q} = \mathbf{L}^{-1}\mathbf{s}$, being $\mathbf{s}$ the vector that gathers the forces in the m branches. The length of the branches $l_i = \sqrt{u_i^2 + v_i^2 + w_i^2}$ are in the square matrix $\mathbf{L} = \text{diag}(\mathbf{l})$. Only gravity loads are considered in this study, that here means self-weight. Vertical point forces are prescribed at the unrestrained nodes through vector $\mathbf{p}_z$.

Due to the introduction of the vector $\mathbf{q}$ and $\mathbf{Q} = \text{diag}(\mathbf{q})$, the equilibrium of the unrestrained nodes is given by a set of linear equations that are uncoupled in the three axes, i.e.:

$$\begin{aligned} &\mathbf{C}^T\mathbf{Q}\mathbf{C}\mathbf{x} + \mathbf{C}^T\mathbf{Q}\mathbf{C}_f\mathbf{x}_f = \mathbf{0}, \\ &\mathbf{C}^T\mathbf{Q}\mathbf{C}\mathbf{y} + \mathbf{C}^T\mathbf{Q}\mathbf{C}_f\mathbf{y}_f = \mathbf{0}, \\ &\mathbf{C}^T\mathbf{Q}\mathbf{C}\mathbf{z} + \mathbf{C}^T\mathbf{Q}\mathbf{C}_f\mathbf{z}_f = \mathbf{p}_z. \end{aligned} \tag{2}$$

Within the considered probabilistic framework, the vector $\mathbf{p}_z$ gathers the average value of the point loads, herein the deterministic weight of the bars falling in the tributary area. For any set of design parameters $\mathbf{q}$, the shape of the structure is defined by Eqn. (2). Hence, N load vectors $\mathbf{p}_{zh}$ are taken into account for the statistical samples, and the mean compliance is evaluated as:

$$\mathcal{C}_m = \frac{1}{N}\sum_{h=1}^{N} \mathbf{d}_{Lh}^T\mathbf{K}_L\mathbf{d}_{Lh}, \tag{3}$$

where the nodal displacement vector for the h statistical sample $\mathbf{d}_{Lh}$ may be computed as:

$$\mathbf{K}_L\mathbf{d}_{Lh} = \mathbf{p}_h, \tag{4}$$

being $\mathbf{K}_L(\mathbf{q})$ the global stiffness matrix as a function of the design parameters $\mathbf{q}$, see [11] and [10], and $\mathbf{p}_h = [\mathbf{0}\,\mathbf{0}\,\mathbf{p}_{zh}]^T$.

3 Optimization problem

A multi-constrained minimization problem is stated in terms of any set of force densities $\mathbf{q}$:

$$
\begin{cases}
\min_{\mathbf{q}} \ \mathcal{C}_m = \dfrac{1}{N} \displaystyle\sum_{h=1}^{N} \mathbf{d}_{Lh}^T \mathbf{K}_L \mathbf{d}_{Lh} & \text{(5a)} \\
\text{s.t.} \quad \mathbf{C}_x^T \mathbf{Q} \mathbf{C}_x \mathbf{x} + \mathbf{C}_x^T \mathbf{Q} \mathbf{C}_{fx} \mathbf{x}_f = \mathbf{0}, & \\
\qquad \mathbf{C}_y^T \mathbf{Q} \mathbf{C}_y \mathbf{y} + \mathbf{C}_y^T \mathbf{Q} \mathbf{C}_{fy} \mathbf{y}_f = \mathbf{0}, & \text{(5b)} \\
\qquad \mathbf{C}_z^T \mathbf{Q} \mathbf{C}_z \mathbf{z} + \mathbf{C}_z^T \mathbf{Q} \mathbf{C}_{fz} \mathbf{z}_f = \mathbf{p}_z, & \\
\qquad \mathbf{K}_L \mathbf{d}_{Lh} = \mathbf{p}_h, \quad \text{for} \ h = 1...N, & \text{(5c)} \\
\qquad \left(\sum_{j=1}^{m} l_j - \dfrac{l_{t,min} + l_{t,max}}{2} \right)^2 \leq \left(\dfrac{l_{t,max} - l_{t,min}}{2} \right)^2, & \text{(5d)} \\
\qquad \dfrac{1}{m} \mathbf{l}^T \mathbf{l} - \left(\sum_{j=1}^{m} l_j \right)^2 \leq l_{sd,max}^2. & \text{(5e)}
\end{cases}
$$

In the above statement, the system of Eqn. (5b) prescribes the equilibrium of the unrestrained nodes in the three spatial directions, to compute $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ from $\mathbf{q}$. For that shape, Eqn. (5c) is used to compute $\mathbf{d}_{Lh}$ that are the vectors of nodal displacements that satisfy the elastic equilibrium for the load vectors $\mathbf{p}_h$. The vectors $\mathbf{d}_{Lh}$ allow for evaluating the objective function, i.e. the mean structural compliance $\mathcal{C}_m$ in Eqn. (5a).

Eqns. (5d) and (5e) are geometric constraints. The former is used to prescribe the minimum ($l_{t,min}$) and maximum ($l_{t,max}$) value of the total length of the network. The latter enforces $l_{sd,max}$ as the maximum allowed standard deviation of the bar lengths. The coordinate difference of the connected points given in Eqn. (1) are used to enforce these geometric constraints in a straightforward way.

The multi-constrained minimization problem is solved by means of the Method of Moving Asymptotes [12], see the discussion in Section 1. Since MMA is a first order approach, the sensitivity of the objective function and constraints with respect to the minimization variables in $\mathbf{q}$ is needed, see e.g. [10].

4 Numerical example

The procedure sketched above is applied to the optimal design of a lattice dome, see e.g. [13].

The topology of the so-called Schwedler dome is used as a reference to define the connectivity of the sought networks. This structural scheme includes intersecting ribs, rings, and diagonal elements, see Figure 1. The investigated dome has a radius equal to $r = 6$ m. It is considered that the dome is fabricated using steel profiles of the type CHS 48.3 / 2.6, with weight per unit volume equal to $\gamma = 78.5$ kN/m^3 and Young's modulus $E = 210$ GPa. Figure 1 provides a map of the nodal loads computed for the relevant configuration.

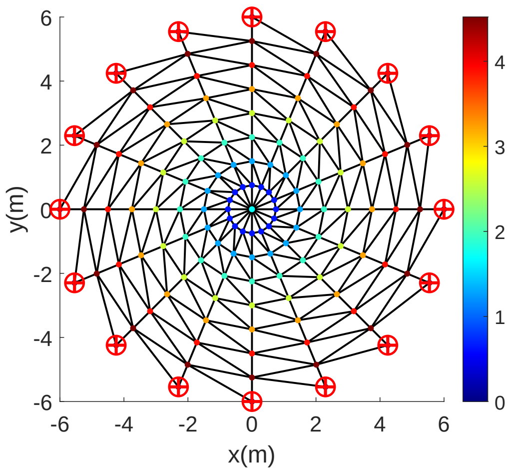


Figure 1: Topology of the grid and relevant nodal loads due to self weight (in kN).

The minimum and the maximum value of the total length of the networks are $l_{min} = 430$ m and $l_{max} = 440$ m, respectively. The parameter $l_{sd,max}$, i.e. the maximum allowed standard deviation of the element lengths, is set to 0.55 m. For all the examples that follow, the force densities are sought in the range -5 kN/m $\leq q_j \leq 5$ kN/m.

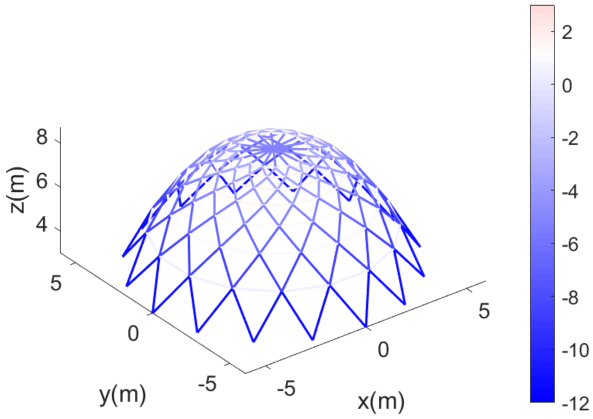
The goal is to investigate the shape of the dome according to Eqn. (5), i.e. finding the set of design parameters $\mathbf{q}$ that maximize the overall stiffness, such that constraints on the total length of the network and on the length of the bars are fulfilled.

It must be remarked that the system in Eqn. (2) is linear if the load vector is design-independent. Hence, an iterative procedure is implemented, by updating the load vector after each optimization run until convergence of results is met.

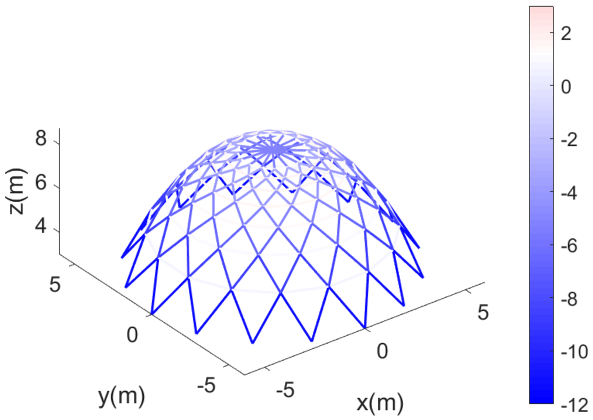
At first, the deterministic case is considered, minimizing the compliance under the average load. Figures 2(a) and 3(a) present the optimal solution along with maps of the elastic forces. The achieved layout is fully feasible with the enforced constraints. In particular, the total length of the dome is equal to the lower bound $l_{min} = 430$ m. It must be remarked that the achieved layout is a major variation with respect to the reference one. An insight on the structural rigidity of pin-jointed space trusses with cyclic symmetry is given in [14].

Then, two additional investigations are performed considering different coefficients of variation (COVs), working with $N = 1000$ random load cases. The result depicted in Figures 2(b) and 3(b) refers to COV=0.10, whereas that in Figures 2(c) and 3(c) refers to COV=0.50. The relevant force maps refer to the elastic equilibrium for the average value of the load.

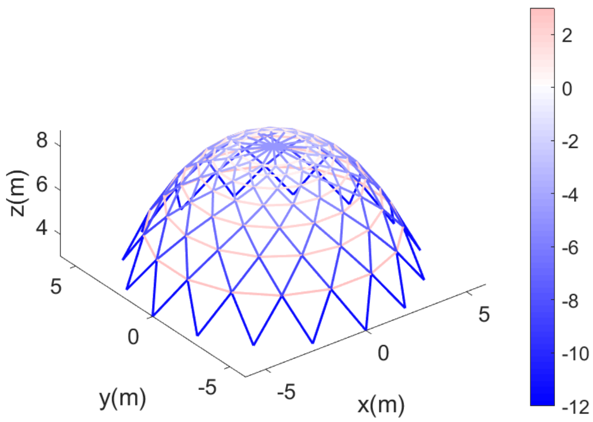
The former solution is almost the same as that found in case of deterministic load. The value of the objective function at convergence is only 1% larger. This means that the statically indeterminate network sought within a deterministic framework of minimum compliance performs well when a minor uncertainty has to be tackled.



(a)



(b)



(c)

Figure 2: Optimal network and elastic forces in the bar (in kN) under average load for different COVs (3D views): 0 (a), 0.10 (b), 0.50 (c).

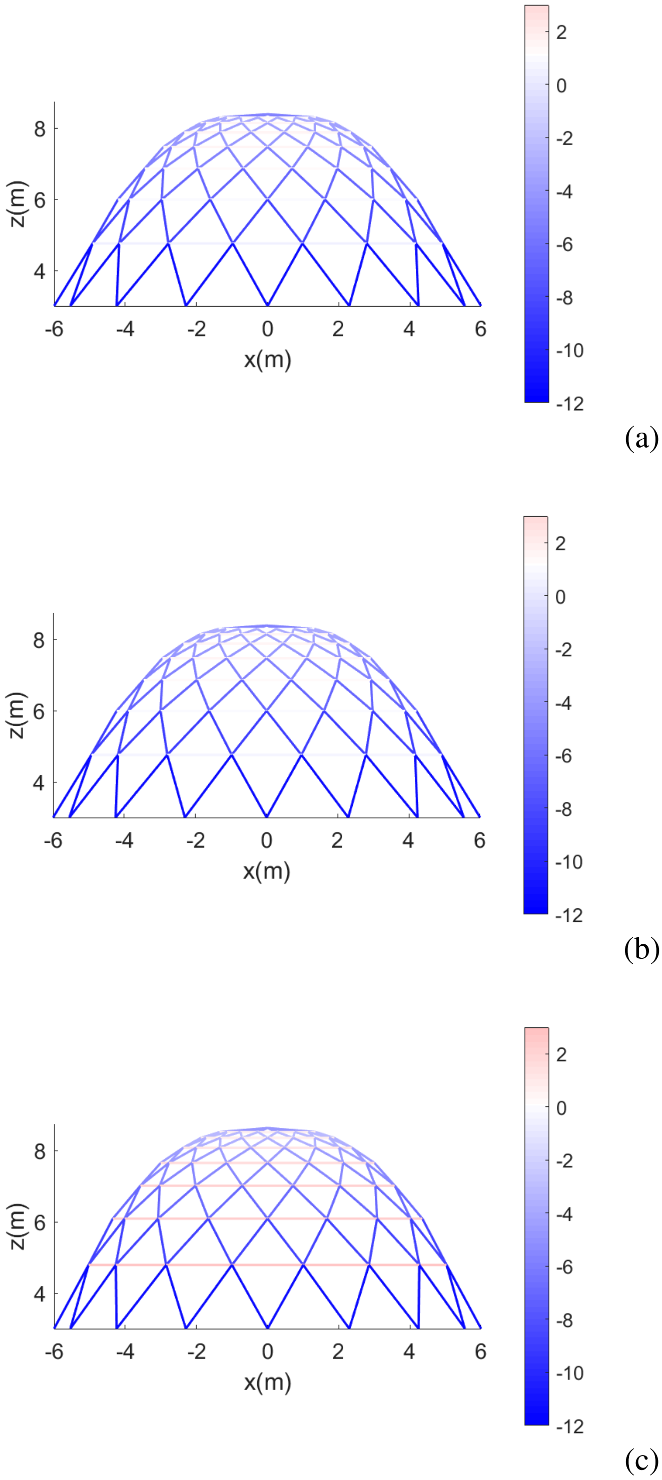


Figure 3: Optimal network and elastic forces in the bar (in kN) under average load for different COVs (side views): 0 (a), 0.10 (b), 0.50 (c).

The latter solution preserves the topology but with different total length (now equal to the upper bound $l_{max} = 440\,\mathrm{m}$) and higher elevation of the top layer, see in particular Figure 3. The value of the objective function at convergence is 25% larger than in the deterministic framework. Looking at the map of the forces acting in the parallels under the effect of the average load, it may be noticed that larger tensile hoop forces exist than in the previous cases. This means that the statics of the optimal dome may be remarkably affected by loading uncertainties.

5 Concluding remarks

The optimal design of reticulated shells has been dealt with, exploring optimal solutions that can be retrieved by a form-finding approach considering loading uncertainties. The force density method has been implemented to address the equilibrium of reticulated shells whose branches are required to behave as struts either ties. Optimal networks have been sought by coupling the equilibrium-based form-finding procedure to the control of the elastic strain energy under the effect of loading uncertainties. A Monte Carlo approach has been used to generate statistical samples and the mean compliance computed across the load cases has been adopted as objective function to be minimized. Constraints on the total length of the network and on the standard deviation of the length of its members have been enforced. Techniques of sequential convex programming that were originally conceived to handle formulations of size optimization for elastic structures have been adopted to attack the arising multi-constrained problem. Funicular networks that are fully feasible with respect to the set of enforcements have been retrieved, exploring optimal solutions for a dome where uncertainties affect self-weight.

Preliminary results confirm that the layout of the dome is sensitive to loading uncertainties. Indeed, the ongoing research is focused on the adoption of probabilistic constraints, see e.g. [15], to investigate optimal solutions depending on the accepted probability of failure.

Acknowledgements

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