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# A Pure Lagrangian Formulation of a Hydroacoustic Fluid-Structure Problem for the Simulation of Underwater Transducers

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## Abstract

The numerical simulation of the small acoustic perturbations produced by an underwater transducer must involve convected models to capture the interaction of the wave propagation phenomena and the underlying fluid motion. Obviously, the spatial location of the transducer is also affected by the fluid motion, and simultaneously, the physical region of interest will vary over time due to tidal and gravity waves. Both fluid-structure phenomena, the acoustic interaction of the transducer with the surrounding fluid motion, and the underlying coupled hydrodynamic phenomena have very different time and spatial scales, and they must be solved simultaneously. This work presents a novel approach where both time scales are considered separately, and both fluid-structure problems are solved numerically in a pure Lagrangian formulation based on velocity and displacement fields by finite element procedures using two different meshes. Galbrun's model is used to compute the time-harmonic acoustic response of the underwater transducer. In contrast, the underlying fluid motion and the position of the transducer, which is considered a rigid solid, are computed implicitly using an implicit Newmark's time marching scheme. Some numerical benchmarks in different scenarios are provided to illustrate the features of the proposed numerical method.

**Keywords:** Lagrangian formulation, fluid-structure problems, Navier-Stokes equations, rigid solid, underwater transducers, Galbrun's model, linear high-order schemes, finite element method

# 1 Introduction

The numerical simulation of fluid-structure problems in underwater environments is a very challenging task mainly because it involves a variety of coupled physical phenomena, and they occur at very different times and spatial scales. This complexity is especially relevant to the acoustic propagation of signals produced by underwater transducers, typically used for monitoring and mapping the seabed of coastal areas [1], where the typical complexity of the fluid dynamics computations have to be tackled in combination with the numerical prediction of acoustic wave propagation phenomena.

The analysis of fluid-structure interaction problems has been studied in the scientific literature profusely, not only in Eulerian, Lagrangian, or Arbitrary Eulerian-Lagrangian formulations [2] and the numerical simulation of these problems have been addressed using different numerical methods, such as finite element [3] or finite volume method [4] among others. More precisely, different works have dealt with the numerical simulation of the motion of a rigid obstacle in the presence of an underlying fluid [5].

This work presents a pure Lagrangian formulation of a hydroacoustic fluid-structure problem to simulate the interaction. This material formulation makes it possible to rewrite the fluid-structure problem in two stages, using only a one-way weak coupling between the hydrodynamic and the acoustic model. In this manner, the underlying fluid motion, and, namely, the deformed material configuration of the fluid domain, is computed using Newmark's implicit time marching scheme in a coarse mesh (see [6] for more details). This deformed Lagrangian configuration is used to linearize the acoustic model and derive Galbrun's governing equation [7], whose convected terms are time-dependent but considered fixed at the high-frequency timescale of the wave propagation phenomena. A finer mesh, depending on the wavelength of the acoustic wave, is used to compute the acoustic response of the underwater transducer, which is considered a rigid solid floating on the free surface of the fluid.

The structure of the paper is as follows. In Section 2, the fluid-structure problem is formulated in Lagrangian coordinates. In Section 3, Galbrun's model for the acoustic problem is presented. The spatial and time discretization is described in Section 4. Then, Section 5 includes a numerical benchmark is presented to illustrate the performance of the proposed method. Finally, in Section 6, some concluding remarks are presented.

# 2 Coupled fluid-structure model in Lagrangian coordinates

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\Gamma$. Let us assume that $\Gamma$ is divided into three disjoint parts: $\Gamma = \Gamma^D \cup \Gamma^N \cup \Gamma^R$, where the superscript capital letters refer to rigid boundary (or prescribed velocity) on $\Gamma^D$, free load boundary $\Gamma^N$, rigid solid boundary in contact with the fluid domain (wet surface) on $\Gamma^R$.

Let $\mathsf{X} : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}^2$ be a *motion* in the sense of Gurtin [8]. In particular, $\mathsf{X} \in \mathbf{C}^3(\overline{\Omega} \times \mathbb{R})$ and for each fixed $t \in \mathbb{R}$, $\mathsf{X}(\cdot, t)$ is a one-to-one function satisfying $\det \mathbf{F} > 0$ in $\overline{\Omega} \times \mathbb{R}$, being $\mathbf{F}(\cdot, t)$ the Jacobian tensor of the deformation $\mathsf{X}(\cdot, t)$. For given $\mathcal{A} \subset \overline{\Omega}$, we denote $\mathcal{A}_t := \mathsf{X}(\mathcal{A}, t)$. In practice, a bounded time interval is considered for the motion, namely, $[t_0, t_f]$, being $t_0$, $t_f$ two non-negative numbers. For simplicity, in this work, it is assumed $\mathsf{X}(\mathsf{p}, t_0) = \mathsf{p}$ for all $\mathsf{p} \in \overline{\Omega}$.

Let us introduce the *trajectory* of the fluid motion $\mathcal{T} := \{(\mathsf{x}, t) : \mathsf{x} \in \overline{\Omega}_t,\ t \in [t_0, t_f]\}$. If $\Psi$ is a spatial (Eulerian) field, its *material* (Lagrangian) description $\Psi_m$ is defined by

$$\Psi_m(\mathsf{p}, t) := \Psi(\mathsf{X}(\mathsf{p}, t), t), \quad \forall (\mathsf{p}, t) \in \overline{\Omega} \times [t_0, t_f]. \tag{1}$$

For clarity in the exposition, those expressions involving space and time derivatives will be denoted following the conventions of Gurtin's monograph [8]. In particular, if $\Psi$ is a smooth spatial vector field, $\dot{\Psi}$ denotes the *material time derivative* with respect to time, that is

$$\dot{\Psi}(\mathsf{x}, t) = \frac{\partial}{\partial t} \left(\Psi(\mathsf{X}(\mathsf{p}, t), t)\right)_{|\mathsf{p}=\mathsf{P}(\mathsf{x},t)} = \Psi'(\mathsf{x}, t) + \operatorname{grad}_{\mathsf{x}} \Psi(\mathsf{x}, t) \mathbf{v}(\mathsf{x}, t),$$

being the last equality obtained by using the chain rule.

We consider a viscous, incompressible Newtonian fluid in a time-dependent domain which may present large deformations. Then, the Eulerian governing equations for the fluid are the unsteady incompressible Navier-Stokes equations. More precisely, we consider the following initial-boundary value problem (motion equation of an incompressible Newtonian fluid): Find two functions $\mathbf{v} : \mathcal{T} \longrightarrow \mathbb{R}^2$ and $\pi : \mathcal{T} \longrightarrow \mathbb{R}$ such that

$$\rho \mathbf{v}' + \rho \operatorname{grad} \mathbf{v} \mathbf{v} - \operatorname{div} \sigma = \mathbf{b} \qquad \text{in } \mathcal{T}, \tag{2}$$

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } \mathcal{T}, \tag{3}$$

where the stress tensor is given by $\sigma = -\pi \mathbf{I} + \mu(\operatorname{grad} \mathbf{v} + \operatorname{grad} \mathbf{v}^t)$, subject to the boundary conditions

$$\mathbf{v}(\mathsf{x}, t) = \mathbf{v}_D(\mathsf{x}, t) \qquad \text{on } \Gamma_t^D, \tag{4}$$

$$\mathbf{v}(\mathsf{x}, t) = \boldsymbol{\alpha}'_R(t) + \boldsymbol{\omega}_R(t) \times (\mathsf{x} - \boldsymbol{\alpha}_R(t)) \qquad \text{on } \Gamma_t^R, \tag{5}$$

$$\sigma(\mathsf{x}, t)\mathbf{n}(\mathsf{x}, t) = \mathbf{h}(\mathsf{x}, t) \qquad \text{on } \Gamma_t^N, \tag{6}$$

where $\mathbf{n}$ is the outward unit normal vector to $\Gamma_t^N$, $\boldsymbol{\omega}_R(t)$ is the angular velocity of the centre of mass $\boldsymbol{\alpha}_R(t)$ of the solid, which satisfies the rigid solid governing equations

$$m_R \boldsymbol{\alpha}''_R = m_R \mathbf{b}_R + \mathbf{F}_R \qquad \text{in } (t_0, t_f), \tag{7}$$

$$\mathbf{I}_R \boldsymbol{\omega}'_R + \boldsymbol{\omega}_R \times \mathbf{I}_R \boldsymbol{\omega}_R = \mathbf{T}_R \qquad \text{in } (t_0, t_f). \tag{8}$$

The problem is completed by the following initial conditions:

$$\mathbf{v}(\cdot, t_0) = \mathbf{v}^0 \qquad \text{in } \overline{\Omega}, \tag{9}$$

$$\boldsymbol{\alpha}_R(t_0) = \boldsymbol{\alpha}_R^0, \quad \boldsymbol{\alpha}'_R(t_0) = \boldsymbol{\alpha}_R^1, \quad \boldsymbol{\omega}_R(t_0) = \boldsymbol{\omega}_R^0. \tag{10}$$

In the above equations, the mass density $\rho : \mathcal{T} \longrightarrow \mathbb{R}$, the dynamic viscosity $\mu : \mathcal{T} \longrightarrow \mathbb{R}$, the volume force $\mathbf{b} : \mathcal{T} \longrightarrow \mathbb{R}^2$, the prescribed boundary velocity $\mathbf{v}_D(\cdot, t) : \Gamma_t^D \longrightarrow \mathbb{R}^2$ and the prescribed boundary traction $\mathbf{h}(\cdot, t) : \Gamma_t^N \longrightarrow \mathbb{R}^2$, $t \in (t_0, t_f)$, are given spatial fields, $\mathbf{I}$ is the identity second order tensor and $\mathbf{n}(\cdot, t)$ is the outward unit normal vector to $\Gamma_t$. Regarding the rigid solid equations, $m_R$ is the mass of the rigid solid, $\mathbf{I}_R(t)$ is its inertia tensor with respect to $\boldsymbol{\alpha}_R(t)$, $\mathbf{b}_R(t)$ is the volume force acting on the solid, the external hydrodynamic force is given by

$$\mathbf{F}_R(t) = -\int_{\Gamma_t^R} \sigma(\mathsf{x}, t)\mathbf{n}(\mathsf{x}, t)\, dS_{\mathsf{x}},$$

and the external torque is given by

$$\mathbf{T}_R(t) = -\int_{\Gamma_t^R} (\mathsf{x} - \boldsymbol{\alpha}_R(t)) \times \sigma(\mathsf{x}, t)\mathbf{n}(\mathsf{x}, t)\, dS_{\mathsf{x}}.$$

Notice that Equations (2)-(10) are expressed in spatial coordinates, $\mathsf{x} = \mathbf{X}(\mathsf{p}, t)$, belonging, in general, to an unknown domain. In order to avoid this difficulty, the Eulerian model can be rewritten in the known reference configuration $\overline{\Omega}$. In this manner, since any rigid motion $\mathsf{X}_R$ can be written in Lagrangian coordinates [8] as

$$\mathsf{X}_R(\mathsf{p}, t) = \boldsymbol{\alpha}_0 + \mathbf{Q}_R(t)(\mathsf{p} - \boldsymbol{\alpha}_0),$$

where $\boldsymbol{\alpha}_0$ is the centre of gravity at the reference configuration and $\mathbf{Q}(t)$ is its associated rotation matrix (determined by the Euler's angles of the rigid motion), then, the Lagrangian model is stated as follows: Find two functions $\mathbf{v}_m : \overline{\Omega} \times [t_0, t_f] \longrightarrow \mathbb{R}^2$ and $\pi_m : \overline{\Omega} \times [t_0, t_f] \longrightarrow \mathbb{R}$ satisfying

$$\rho_m \dot{\mathbf{v}}_m - \frac{1}{\det \mathbf{F}} \operatorname{Div} \left(\sigma_m \det \mathbf{F}\mathbf{F}^{-t}\right) = \mathbf{b}_m \qquad \text{in } \Omega \times (t_0, t_f), \tag{11}$$

$$\nabla \mathbf{v}_m \cdot \mathbf{F}^{-t} = 0 \qquad \text{in } \Omega \times (t_0, t_f), \tag{12}$$

where $\sigma_m = -\pi_m \mathbf{I} + \mu_m \left(\nabla \mathbf{v}_m \mathbf{F}^{-1} + \mathbf{F}^{-t} \left(\nabla \mathbf{v}_m\right)^t\right)$ is the material description of the Cauchy stress tensor. The Lagrangian fluid model is completed with the boundary and initial conditions:

$$\mathbf{v}_m = (\mathbf{v}_D)_m \ \text{ on } \Gamma^D \times [t_0, t_f], \tag{13}$$

$$\mathbf{v}_m = \dot{\mathsf{X}}_R = \dot{\mathbf{Q}}_R(t)(\mathsf{p} - \boldsymbol{\alpha}_0) \ \text{ on } \Gamma^R \times [t_0, t_f], \tag{14}$$

$$\sigma_m \mathbf{F}^{-t}\mathbf{m} = |\mathbf{F}^{-t}\mathbf{m}|\mathbf{h}_m \ \text{ on } \Gamma^N \times [t_0, t_f], \tag{15}$$

$$\mathbf{v}_m(\cdot, t_0) = \mathbf{v}^0 \ \text{ in } \overline{\Omega}, \tag{16}$$

where $\mathbf{m}$ is the outward unit normal vector to $\partial\Omega$. The Lagrangian description of the rigid solid governing convected equations is given by

$$\mathbf{I}_{Rm}\dot{\hat{\boldsymbol{\omega}}}_R + \hat{\boldsymbol{\omega}}_R \times \mathbf{I}_R \hat{\boldsymbol{\omega}}_R = \mathbf{Q}_R^t \mathbf{T}_R \qquad \text{in } (t_0, t_f),$$

$$\dot{\mathbf{Q}}_R = \mathbf{Q}_R \hat{\mathbf{W}}_R \qquad \text{in } (t_0, t_f),$$

being $\hat{\mathbf{W}}_R$ the convected spin of the rigid motion, represented by a skew matrix with axial-vector $\hat{\boldsymbol{\omega}}_R$. Notice that in this Lagrangian formulation, all the coefficients of the ordinary differential equations governing the rigid motion are constant (time-independent). The corresponding initial conditions for this convected formulation of the rigid motion are $\hat{\boldsymbol{\omega}}_R(t_0) = (\mathbf{Q}_R^0)^t \boldsymbol{\omega}_R^0$ and $\mathbf{Q}_R(t_0) = \mathbf{Q}_R^0$.

# 3 Modelling small perturbations from an underlying flow

Once the coupled fluid-structure model has been written in Eulerian and Lagrangian coordinates, a linear model governing the small perturbations from an underlying motion (possibly with a different time and spatial scale) can be derived. The underlying given motion is denoted by $\mathsf{Y}$ and by $\mathsf{X}$ accounts for the motion associated with a small isentropic perturbation of it (see [9] for a detailed discussion). In what follows, different subscripts account for physical quantities associated with a given motion.

With this aim, the small perturbation motion $\mathsf{X}$ must be read as the composition of two different motions: the motion $\mathsf{Y}$ and the motion $\mathsf{Z} : \mathcal{T}_{\mathsf{Y}} \to \mathbb{R}^2$ defined by

$$\mathsf{Z}(\mathsf{y}, t) = \mathsf{X}\left(\mathsf{P}_{\mathsf{Y}}(\mathsf{y}, t), t\right) \qquad \forall (\mathsf{y}, t) \in \mathcal{T}_{\mathsf{Y}}, \tag{17}$$

where $\mathsf{P}_{\mathsf{Y}}$ is the reference application of motion $\mathsf{Y}$ (its inverse at a fixed time $t$). Hence, motion $\mathsf{Z}$ is the motion $\mathsf{X}$ rewritten in terms of the Eulerian coordinates associated with the motion $\mathsf{Y}$. From (17), if we rewrite it in terms of material points using $\mathsf{y} = \mathsf{Y}(\mathsf{p}, t)$ and $\mathsf{p} = \mathsf{P}_{\mathsf{Y}}(\mathsf{y}, t)$, we have

$$\mathsf{X}(\mathsf{p}, t) = \mathsf{Z}(\mathsf{Y}(\mathsf{p}, t), t) \qquad \forall (\mathsf{p}, t) \in \Omega \times [t_0, t_f].$$

and hence, computing the gradients of both sides of the equation above leads to

$$\mathbf{F}_{\mathsf{X}}(\mathsf{p}, t) = \mathbf{F}_{\mathsf{Z}}(\mathsf{Y}(\mathsf{p}, t), t)\mathbf{F}_{\mathsf{Y}}(\mathsf{p}, t), \tag{18}$$

where $\mathbf{F}_{\mathsf{Z}} = \mathrm{grad}_{\mathsf{y}}(\mathsf{Z})$. In addition, since the motion $\mathsf{X}$ is considered as a small perturbation from the motion described by $\mathsf{Y}$, the difference in the spatial position given by both motions should be small. Hence, it is natural to introduce the vector field $\mathbf{w}$ of the perturbed displacement as a function of Eulerian variables of motion $\mathsf{Y}$ given by

$$\mathbf{w}(\mathsf{y}, t) = \mathsf{Z}(\mathsf{y}, t) - \mathsf{y}, \tag{19}$$

and its material description $\mathbf{w}_m$ in material (Lagrangian) coordinates which satisfies

$$\mathbf{w}_m(\mathsf{p}, t) = \mathbf{w}(\mathsf{Y}(\mathsf{p}, t), t) = \mathsf{Z}(\mathsf{Y}(\mathsf{p}, t), t) - \mathsf{Y}(\mathsf{p}, t) = \mathsf{X}(\mathsf{p}, t) - \mathsf{Y}(\mathsf{p}, t), \tag{20}$$

and also

$$\dot{\mathbf{w}}(\mathsf{y}, t) = \frac{\partial \mathsf{X}}{\partial t}\left(\mathsf{P}_{\mathsf{Y}}(\mathsf{y}, t), t\right) - \mathbf{v}_{\mathsf{Y}}(\mathsf{y}, t) = \mathbf{v}_{\mathsf{X}}(Z(\mathsf{y}, t), t) - \mathbf{v}_{\mathsf{Y}}(\mathsf{y}, t),$$

for all $(\mathsf{y}, t) \in \mathcal{T}_{\mathsf{Y}}$, and where $\dot{\mathbf{w}}$ is the material time derivative of $\mathbf{w}$ with respect to motion $\mathsf{Y}$.

With these relations about the motions in mind, if the following three hypotheses are considered: (i) the fluid is compressible, (ii) the viscosity effects can be neglected, and (iii) the pressure field $\pi_{\mathsf{Y}}$ admits a response function in terms of the mass density $\rho_{\mathsf{Y}}$ and the entropy $s_{\mathsf{Y}}$, this is, $\pi_{\mathsf{Y}} = \hat{\pi}\left(\rho_{\mathsf{Y}}, s_{\mathsf{Y}}\right)$, then, the balance of forces and momentum in Lagrangian coordinates are taken into account, rewritten using the change of variable $\mathsf{y} = \mathsf{Y}(\mathsf{p}, t)$, the second term in the left-hand side of (11) is given by

$$\begin{aligned} &\det \mathbf{F}_{\mathsf{Z}} \hat{\pi}\left(\rho_{\mathsf{X}}(\mathsf{Z}, t), s_{\mathsf{Y}}\right) \mathbf{F}_{\mathsf{Z}}^{-t} - \hat{\pi}\left(\rho_{\mathsf{Y}}, s_{\mathsf{Y}}\right) \mathbf{I} \\ &= -\rho_{\mathsf{Y}} \frac{\partial \hat{\pi}}{\partial \rho}\left(\rho_{\mathsf{Y}}, s_{\mathsf{Y}}\right) \operatorname{div} \mathbf{w} \mathbf{I} + \operatorname{div} \mathbf{w} \hat{\pi}\left(\rho_{\mathsf{Y}}, s_{\mathsf{Y}}\right) \mathbf{I} - \hat{\pi}\left(\rho_{\mathsf{Y}}, s_{\mathsf{Y}}\right) \operatorname{grad} \mathbf{w}^{t} + o(\operatorname{grad} \mathbf{w}). \end{aligned}$$

After some straightforward algebraic and differential computations, the linearized motion equation from (11)-(12) written in the deformed configuration by the underlying motion $\mathsf{Y}$ is given by

$$\rho_{\mathsf{Y}} \ddot{\mathbf{w}} = -\operatorname{div} \mathbf{w} \operatorname{grad} \pi_{\mathsf{Y}} + \operatorname{grad} \mathbf{w}^{t} \operatorname{grad} \pi_{\mathsf{Y}} + \operatorname{grad}\left(\rho_{\mathsf{Y}} c_{\mathsf{Y}}^{2} \operatorname{div} \mathbf{w}\right) \quad \text{in } \mathcal{T}_{\mathsf{Y}}, \qquad (21)$$

where all the differential operators must be understood in terms of the spatial variable $\mathsf{y}$, and $c_{\mathsf{Y}}$ denotes the fluid speed of propagation at the deformed configuration in the trajectory $\mathcal{T}_{\mathsf{Y}}$.

# 4 Spatial and time discretization

Taking into account the models stated in the sections above, the solution of the coupled problem can be split into two stages: firstly, the underlying motion $\mathsf{Y}$ is computed using the pure Lagrangian non-linear formulation described in Section 2. Once this motion is computed, the deformed configuration $\Omega_{\mathsf{Y}}$, the mass density $\rho_{\mathsf{Y}}$ and the pressure field $\pi_{\mathsf{Y}}$ (involved in the convected terms of the Galbrun's model) can be computed. Since the time scales of the motion $\mathsf{Y}$ and the perturbation displacement $\mathbf{w}$ are drastically different, Galbrun's model can be solved under the time-harmonic assumption because its time scale is several orders of magnitude smaller than the scale of the motion $\mathsf{Y}$.

The time-dependent non-linear hydrodynamic problem has been solved numerically using a linear second-order Lagrangian scheme. In contrast, the acoustic problem has been solved using a wavelength-dependent refined mesh. More precisely, the fluid problem in Lagrangian coordinates has been discretized by using a linearized version of the non-linear optimal accuracy Newmark method compatible with unconditional stability. This linearized Newmark algorithm preserving the accuracy properties of the corresponding non-linear Newmark scheme has been proposed in [6]. Concerning the spatial discretization of fluid problem, we have used a stable combination of continuous finite element spaces on triangular meshes. More precisely, we consider the so-called mini-element (continuous piecewise-linear+bubble for the velocity and continuous piecewise-linear for pressure).

One of the main difficulties in working with a two-scale problem using two different meshes is the communication between the two problems. To transfer the hydrodynamic data, namely the hydrostatic pressure field, into the acoustic mesh. First, the hydrodynamic velocity field is interpolated into the acoustic mesh using the nearest neighbour interpolation procedure under a piecewise continuous linear finite element on the mesh. Then, the hydrostatic pressure field $\pi_{\mathrm{Y}}$ is computed from the interpolated velocity field, and the gradient of the pressure field is assumed in the Raviart-Thomas space, where originally the displacement field solution of the Galbrun's model is computed.

# 5 Numerical benchmark

In this section, we present a numerical benchmark, which mimics a wave tank facility, to illustrate the features of the proposed method. We consider a initial two-dimensional domain $\Omega = [0, 5] \times [0, 0.5]$ m filled completely with an incompressible fluid. The dynamic viscosity is $\mu = 0.001\,\mathrm{Ns/m^2}$, $\rho = 1\,\mathrm{kg/m^3}$, and null volume source, $\mathbf{b} = \mathbf{0}$. The analytical velocity $\mathbf{v}_D = (v_1, v_2)$ that has been used to impose the Dirichlet boundary condition on the boundary of the domain is obtained from the linear potential flow theory, namely,

$$v_1(p_1, p_2, t) = A\omega \frac{\cosh(k(p_2 + h))}{\sinh(kh)} \cos(kp_1 - \omega t),$$

$$v_2(p_1, p_2, t) = A\omega \frac{\sinh(k(p_2 + h))}{\sinh(kh)} \sin(kp_1 - \omega t),$$

being $h = 0.5\,\mathrm{m}$ the initial depth, $A = 0.005\,\mathrm{m}$ the amplitude, $k = 10\,\mathrm{m^{-1}}$ the wave number and $\omega$ the angular frequency satisfying the dispersion equation $\omega^2 = gk\tanh(kh)$, being $g$ the gravity acceleration.

In this simple scenario, the size of the underwater transducer is $0.02\,\mathrm{m}$. Hence, it is assumed that it does not affect the underlying motion of the fluid and that it is floating on the free surface of the physical fluid domain. It is located in centred at $p_1 = 2.5\,\mathrm{m}$. Additionally, since it is located at the free surface of the fluid domain, its rigid motion is determined by the normal orientation and amplitude of the fluid motion at its location. To compute the proposed hybrid two time-scale simulations, first, the hydrodynamic problem has been computed using the mesh depicted in Figure 1. The hydrodynamic velocity field and its modulus are depicted in Figures 2 and 3.

Once the hydrodynamic problem has been solved, the acoustic problem has been computed using the mesh depicted in Figure 4. Notice that the acoustic mesh has to be rebuilt at each time step of the hydrodynamic problem since the Lagrangian configuration is time-dependent. The pressure field is depicted in Figure 6 and its associated velocity field in Figure 5.

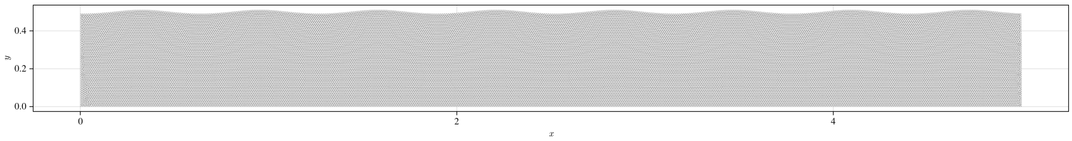


Figure 1: Deformed mesh at $t = 0.34$ s for the hydrodynamic problem.

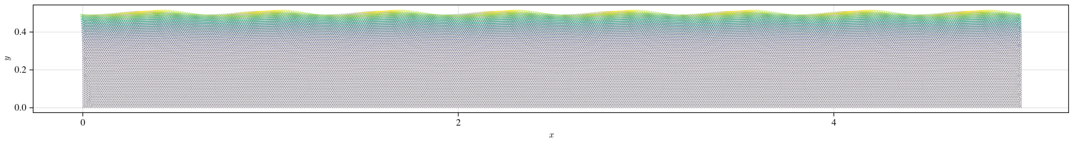


Figure 2: Velocity field at $t = 0.34$ s for the hydrodynamic problem.

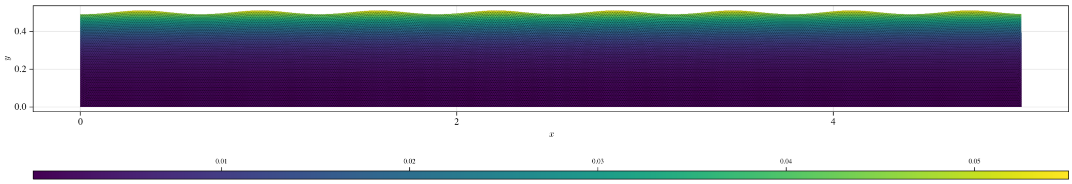


Figure 3: Velocity modulus at $t = 0.34$ s for the hydrodynamic problem.

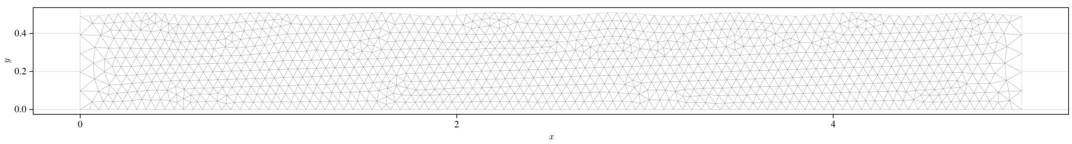


Figure 4: Deformed mesh at $t = 0.34$ s for the acoustic problem.

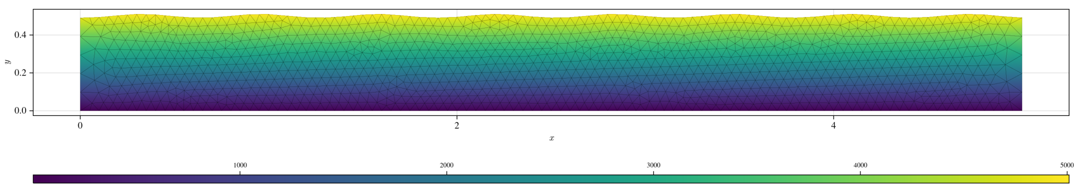


Figure 5: Hydrodynamic pressure field at $t = 0.34$ s for the acoustic problem.

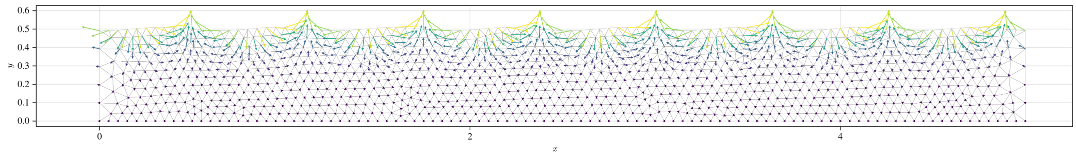

Figure 6: Hydrodynamic velocity field at $t = 0.34\,\mathrm{s}$ for the acoustic problem.

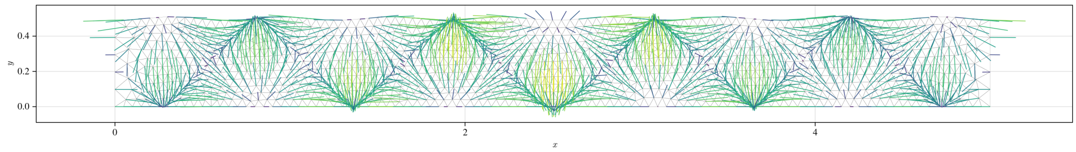

Figure 7: Displacement field at $t = 0.34\,\mathrm{s}$ for the acoustic problem using the Helmholtz's model.

To illustrate the impact of considering Galbrun's model in this numerical test, even in a scenario with a very low Reynolds number, we have computed the acoustic pressure field using Galbrun's model and the classical Helmholtz equation. The numerical results corresponding with the time-harmonic displacement field (vector values and modulus) are depicted in Figures 7-9.

Similarly, the numerical results corresponding with the time-harmonic solutions of the acoustic problem using Galbrun's model are depicted in Figures 10-12. It is observed only a slight difference between the numerical results obtained using the Galbrun's model and the Helmholtz's model. This difference is because the Reynolds number is very low, and the hydrostatic pressure is almost uniform in the shallow water scenario. However, the displacement field on the free surface and also the other boundaries should be computed using Galbrun's model to ensure the correct behaviour of the fluid-structure interaction problem.

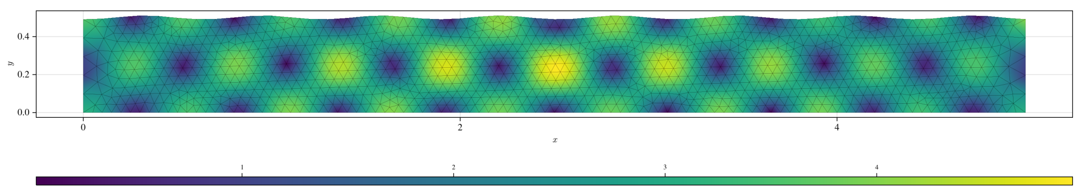

Figure 8: Modulus of the displacement field at $t = 0.34\,\mathrm{s}$ for the acoustic problem using the Helmholtz's model.

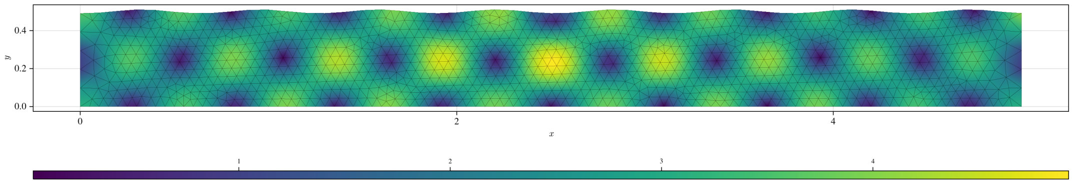

Figure 9: Pressure field at $t = 0.34\,\mathrm{s}$ for the acoustic problem using the Helmholtz's model.

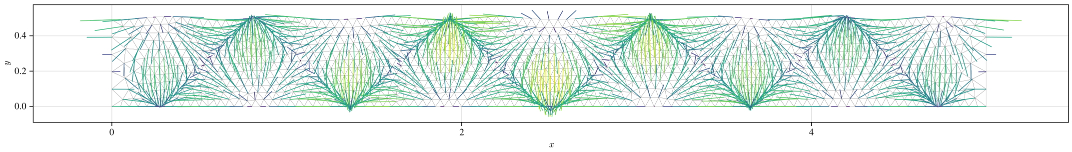

Figure 10: Displacement field at $t = 0.34\,\mathrm{s}$ for the acoustic problem using the Galbrun's model.

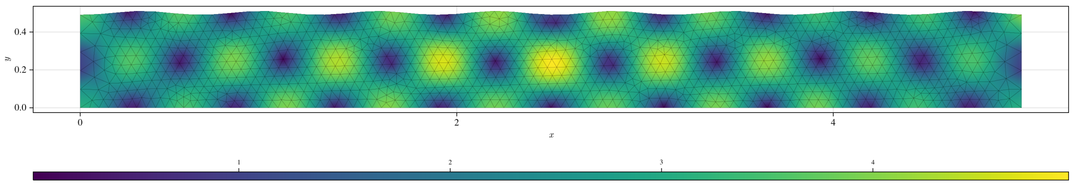

Figure 11: Modulus of the displacement field at $t = 0.34\,\mathrm{s}$ for the acoustic problem using the Galbrun's model.

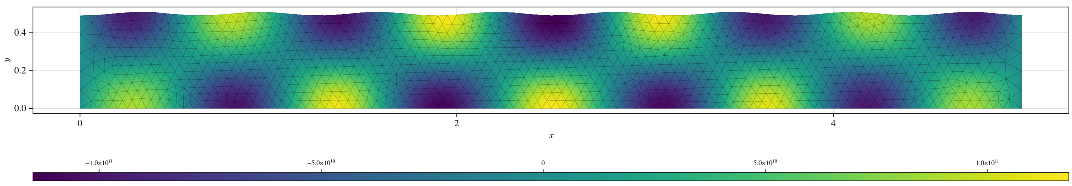

Figure 12: Pressure field at $t = 0.34\,\mathrm{s}$ for the acoustic problem using the Galbrun's model.

# 6 Concluding remarks

This work presents a novel approach to simulating underwater transducers through a pure Lagrangian formulation of a hydroacoustic fluid-structure problem. The study introduces the challenges associated with modelling and simulating underwater transducers, particularly concerning the interaction between the transducer structure and the surrounding fluid.

The proposed formulation is based on the Lagrangian description of fluid motion, which provides a natural framework for treating acoustic fluid-structure interaction problems using Galbrun's model and, hence, the displacement field as a primal physical unknown. This approach allows an efficient computational tool capable of handling two different time and spatial scales simultaneously (treating differently the computation of the underlying motion and the acoustic wave propagation phenomena). The different spatial scales are also considered using two different finite element meshes, one for the hydrodynamic problem and another for the acoustic problem, with specific requirements that attend to the typical wavelength of each physical phenomenon.

Some numerical tests have been presented to illustrate the numerical features of the proposed method. They highlight how a pure Lagrangian formulation with a deformed mesh can ease the linearization of Galbrun's model and the computation of the acoustic displacement field. The differences between Galbrun's and the classical time-harmonic Helmholtz model are also analyzed in this Lagrangian framework to show the impact of the convected terms in the fluid-structure interaction problem.


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