Proceedings of the Fifteenth International Conference on
Computational Structures Technology
Edited by: P. Iványi, J. Kruis and B.H.V. Topping
Civil-Comp Conferences, Volume 9, Paper 11.3
Civil-Comp Press, Edinburgh, United Kingdom, 2024
ISSN: 2753-3239, doi: 10.4203/ccc.9.11.3

Single-Layer Modelling of Semi-Infinite 2D Domains Invoking Periodicity

X. Chen and B. A. Izzuddin

**Department of Civil and Environmental Engineering, Imperial College London
London, United Kingdom**

Abstract

This paper presents an efficient finite element methodology for the static analysis of semi-infinite structures, addressing the common issue in structural engineering where the region under analysis is significantly smaller than its surrounding medium. By dividing the domain into proportionally dimensioned layers of elements and maintaining a consistent or proportional stiffness matrix between layers, the proposed method utilises the finite element analysis of only a single layer for predicting the static response of the entire domain invoking the concept of periodicity. Employing eigenvalue analysis, the method examines the relationship between nodal deflection across different layers and mechanical behaviours including the transient and steady-state response. This facilitates the extraction of recurrent modes, which then characterise the overall static response through modal superposition. The methodology is notably efficient in managing the extensive degrees of freedom typically associated with semi-infinite domains, even when employing a fine mesh at the loaded end to enhance the capturing of local effects. The validity of this method is demonstrated through the analysis of a sample problem involving a hole in an infinite plate, with results consistently verifying the accuracy. Furthermore, a foundation problem is examined, underscoring the broader applicability of the proposed method in infrastructural contexts.

Keywords: finite element method, semi-infinite domain, single-layer modelling, structural periodicity, static analysis, eigenvalue analysis, modal superposition, computation efficiency.

1 Introduction

The analysis of solid mechanics systems with a semi-infinite domain is inherently demanding, yet it is required in numerous structural engineering and soil-structure interaction problems. Particularly challenging are scenarios where boundary effects extend significantly beyond the primary region of interest [1]. Traditional finite element analysis approaches this by extending the mesh considerably and applying fixed displacement or constant stress boundary conditions at the remote boundary. This typically leads to excessive computational demands due to the large number of degrees of freedom (DOFs), especially when employing fine meshes to enhance accuracy. Such traditional numerical method often encounters computational limits, highlighting the need for more efficient modelling techniques.

Several advanced methodologies have been developed to overcome these challenges, including infinite elements method [2], boundary element method [3], wave finite element method [4], and discrete Fourier transform [5]. The infinite element method [2] employs specialised infinite shape functions to map the element geometry from the finite mesh to infinity. An additional decay function is then included in the infinite shape function to allow field variables interpolated at the finite boundary to attenuate as they extend towards infinity. Therefore, the effectiveness of the method depends heavily on the integration of infinite elements into the finite element mesh and the appropriate selection of decay functions for field variables. Although this method reduces nodes in areas remote from the primary zone of interest, it still requires a fine mesh of elements at the finite boundary. The boundary element method [3] applies discretization on the boundary only and therefore reduces the problem dimensionality. The partial differential equations governing the problem are transformed into integral equations that describe how effects propagate from a source point to observation points in the domain using Green's function. The integration equations of field variables, which inherently satisfy the infinite boundary conditions, lead to linear equations at the collocation nodes at infinity to be solved. However, it results in a fully populated matrix representing the system of equations, thus it can be computationally expensive when fine discretisation is employed at the boundary. The most developed method utilising structural periodicity is the wave finite element method [4]. In this method, displacements and forces at the boundaries of each element are coupled, premised on the assumption that a characteristic free wave traverses the periodic system with a propagation constant. Consequently, the analysis necessitates the examination of only a single element. Nonetheless, this method is predominantly employed in the dynamic analysis of structures. Similarly, the discrete Fourier transform [5] uses a complex exponent to map the periodic displacement from an infinite boundary to that of a unit cell, facilitating the study of the static response of an infinite periodic structure. However, this method requires the inversion of the Fourier transform to solve the displacement at internal layers. This inversion is typically performed numerically, and its effectiveness is heavily dependent on the number of approximated integration points. In addition, the complex exponent also implies that the deformation amplitude is only transiently decaying over space with trigonometric oscillation in phase, while the steady-state deformation is ignored.

This paper is aimed at reducing the complexity of static analysis for semi-infinite domains by focusing on a single substructure layer, drawing on some of the above methods that incorporate periodicity. Finite element analysis is confined to the first layer of elements used to represent the subdomain, as illustrated in Figure 1. Mesh periodicity is invoked to reveal similar deformation patterns across each layer of elements. Eigenvalue analysis is applied to recover the static deformation modes that recur periodically or steadily throughout the mesh. This analysis not only addresses transient deformation across space but also examines the potential mechanisms of rigid body translation and uniform tension and compression within the domain, so-called steady-state modes. The method is validated through the analysis of stress concentration in a perforated plate under uniaxial compressing, demonstrating its effectiveness. Additionally, it is applied to a 2D simple foundation problem using axis-symmetric elements. The method is also extendible to model 3D semi-infinite problems, indicating a significant potential for enhanced modelling of a broader range of structural applications.

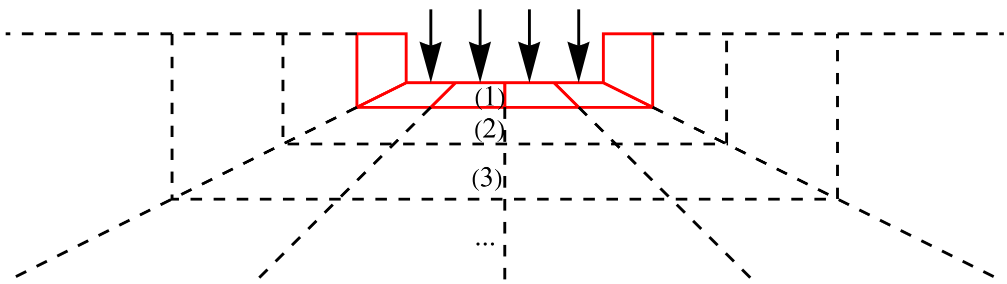


Figure 1: Modelling example of a semi-infinite problem

2 Methods

The problem of a semi-infinite domain can be conceptualised as a structure composed of an infinite array of representative substructures which are meshed uniformly or proportionally and possess an identical number of nodes. In other words, the stiffness matrixes are equal or proportional, depending on their geometric characteristics. By leveraging periodicity, modelling a single layer suffices to describe the mechanical behaviour of the entire structure.

The eigenvalue method recovers the deformation patterns under static loading that exist periodically layer-by-layer. The field deformation is represented as a superposition of those patterns, which are also defined as static modes. According to the transmission characteristics, the modes can be classified into transient modes and steady-state modes. Transient modes capture localised effects stemming from loads and supports, characterised by an initial amplitude that diminishes progressively until it dissipates completely. Conversely, steady-state modes represent a continuous response throughout the space, encapsulating the overall structural behaviour. These modes may manifest constant deformations across layers or vary progressively (e.g. linearly), corresponding to rigid body motions, constant strain modes, etc. While the number of steady-state modes is constant and depends on the specific type of problem, the number of transient modes complements the number of steady-state modes to equal the total number of DOFs for a representative layer. The underlying logic of the method involves embedding the periodic correlation of deformation between layers into the eigenmatrix to extract the transient and steady-state modes.

Figure 2 shows a schematic diagram of two consecutive mesh layers (substructures) with three sets of boundary DOFs. The basic formulation for a substructure (k) is:

$$\begin{bmatrix} \mathbf{K}_{ll} & \mathbf{K}_{lr} \\ \mathbf{K}_{rl} & \mathbf{K}_{rr} \end{bmatrix}^{(k)} \begin{Bmatrix} \mathbf{d}_l^{(k)} \\ \mathbf{d}_r^{(k)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_l^{(k)} \\ \mathbf{f}_r^{(k)} \end{Bmatrix} \tag{1}$$

where $\mathbf{K}$, $\mathbf{d}$, and $\mathbf{f}$ are the stiffness matrix, displacement vector and force vector, respectively. The subscripts l and r denote the left and right sides of the mesh layer. Any internal DOF can be condensed given no external loading is applied to it. Mesh periodicity implies that for two successive layers $\mathbf{K}^{(k+1)} = \alpha \mathbf{K}^{(k)}$, where α is a proportionality factor. If the element geometry is enlarged by a factor of β between layers, it can be shown that $\alpha = 1$ for 2D plane stress/plane strain problems, while $\alpha = \beta$ for axisymmetric problems.

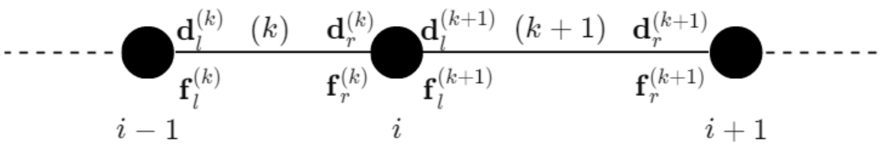


Figure 2: Schematic representation of two consecutive mesh layers of a semi-infinite domain

The continuity of displacements $\mathbf{d}_r^{(k)} = \mathbf{d}_l^{(k+1)}$ and equilibrium of forces $\mathbf{f}_r^{(k)} + \mathbf{f}_l^{(k+1)} = 0$ must be satisfied on the i^{th} interface so that:

$$\mathbf{K}_{rl}^{(k)} \mathbf{d}_{i-1} + (\mathbf{K}_{rr}^{(k)} + \alpha \mathbf{K}_{ll}^{(k)}) \mathbf{d}_i + \alpha \mathbf{K}_{lr}^{(k)} \mathbf{d}_{i+1} = 0 \tag{2}$$

Based on Bloch's theorem [6], the transient modes are recovered by assuming that the nodes at the consecutive boundaries have proportional displacements, with the amplitude multiplied by a constant propagation ratio λ:

$$\mathbf{d}_{i+1} = \lambda \mathbf{d}_i \tag{3}$$

Combining with Equation (2) leads to the following quadratic eigenproblem:

$$\left[\lambda^2 \times \alpha \mathbf{K}_{lr} + \lambda \times \left(\mathbf{K}_{rr} + \alpha \mathbf{K}_{ll} \right) + \mathbf{K}_{rl} \right] \boldsymbol{\phi}_t = \mathbf{0} \tag{4}$$

If there are n DOFs at the i^{th} interface, solving the eigenproblem gives $2n$ eigenvectors. Linearisation of the quadratic eigenproblem leads to a non-symmetric eigenmatrix so that the eigenvalues (λ) are complex numbers. This indicates that there will be both amplitude changes and trigonometric phase oscillations across layers. For a magnitude of λ smaller than 1, the corresponding transient mode $\boldsymbol{\phi}_t$ decays with increasing layer i, and the deformation is localised at the place of excitation. For the magnitude larger than 1, the influence of $\boldsymbol{\phi}_t$ is concentrated at the other end of mesh. If $\lambda = 1$, there is no transient response in the domain since $\boldsymbol{\phi}_t$ remains constant. Different formulations are needed to recover these non-transient modes, which basically extract the static-steady modes.

Constant modes $\boldsymbol{\phi}_0$ imply that all the layers have the same nodal displacements, and these modes are typically rigid body modes. The constant modes can be recovered in the same manner as transient modes with $\lambda = 1$ by solving the eigensystem of $\mathbf{K}_{cons}$:

$$\left[\alpha\mathbf{K}_{lr} + \left(\mathbf{K}_{rr} + \alpha\mathbf{K}_{ll}\right) + \mathbf{K}_{rl}\right]\boldsymbol{\phi}_0 = \mathbf{K}_{cons}\boldsymbol{\phi}_0 = \mathbf{0} \tag{5}$$

The linear modes describe the scenario where there is a base modal $\boldsymbol{\phi}_1$ and an additional constant deformation $\boldsymbol{\phi}_0 \cdot \boldsymbol{a}$ when it propagates to the next layer, where $\boldsymbol{a}$ accounts for the requisite combination of different constant modes. Due to the linear propagation of displacement over the domain space, the existence of linear modes indicates that the incremental displacement between the initial and infinite boundaries is infinite. To obtain the linear modes, Equation (2) can be transferred to:

$$\mathbf{K}_{cons}\boldsymbol{\phi}_1 + \left[\alpha\mathbf{K}_{ll} + 2\alpha\mathbf{K}_{lr} + \mathbf{K}_{rr}\right]\boldsymbol{\phi}_0 \cdot \boldsymbol{a} = \mathbf{K}_{cons}\boldsymbol{\phi}_1 + \mathbf{K}_{temp}\boldsymbol{\phi}_0 \cdot \boldsymbol{a} = \mathbf{0} \tag{6}$$

where $\mathbf{K}_{temp}$ accounts for the linear combination of the constant modes in the eigensystem. Adding orthogonality conditions between the obtained constant modes $\boldsymbol{\phi}_0$ and any linear mode $\boldsymbol{\phi}_1 + \boldsymbol{\phi}_0 \cdot \boldsymbol{a}$, the eigenvalue problem is formed as:

$$\begin{bmatrix} \mathbf{K}_{cons} & \mathbf{K}_{temp} \\ \boldsymbol{\phi}_0^T & \boldsymbol{\phi}_0^T \cdot \boldsymbol{\phi}_0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\phi}_1 \\ \boldsymbol{a} \end{Bmatrix} = \mathbf{K}_{lin} \begin{Bmatrix} \boldsymbol{\phi}_1 \\ \boldsymbol{a} \end{Bmatrix} = \mathbf{0} \tag{7}$$

Using the propagation law for each type of mode, starting from the displacement generated at the initial interface where external loading is applied, the resulting modal deflection at the i^{th} interface is illustrated in Table 1.

Modes	Loaded interface	i^{th}
Transient	$\boldsymbol{\phi}_t$	$\lambda^i \times \boldsymbol{\phi}_t$
Constant	$\boldsymbol{\phi}_0$	$\boldsymbol{\phi}_0$
Linear	$\boldsymbol{\phi}_1$	$\boldsymbol{\phi}_1 + i\boldsymbol{\phi}_0 \cdot \mathbf{a}$

Table 1: Modes at arbitrary layers

There are $2n$ transient and steady-state modes in total, which correspond to the number of DOFs of a single layer of elements. The static response of a structure can be expressed as a linear combination of these mode shapes. The weight for each mode is calculated by solving n force equilibrium conditions at the loaded end/interface and n displacement compatibility conditions at the remote end/interface. Given this, the displacement, strains, and stresses at any location in the domain can be recovered.

3 Illustrative Examples

3.1 Infinite plate with a circular hole

First, an infinite plate with a central circular hole subjected to uniaxial loading is analysed to verify the results against Kirsch's solution [7]. The influence of the hole on this prestressed domain can be modelled by applying the same stress field to the hole surface which makes it free of tractions. Figure 3 shows a proposed mesh for the symmetrical section of the infinite plate. 4-noded bilinear plane stress elements are used, where each layer comprises 6 elements. Despite the expanding geometry of the elements with $\beta = 1.5$, the stiffness matrices remain constant for all layers. Generally, for a plane stress problem, there are two constant modes which correspond to rigid body translations in the X and Y directions, two linear modes which correspond to the constant stretching in both directions, and a series of transient modes. However, since the symmetry conditions are applied (y/x displacement is restrained along the x/y axis), there are only transient modes in this case.

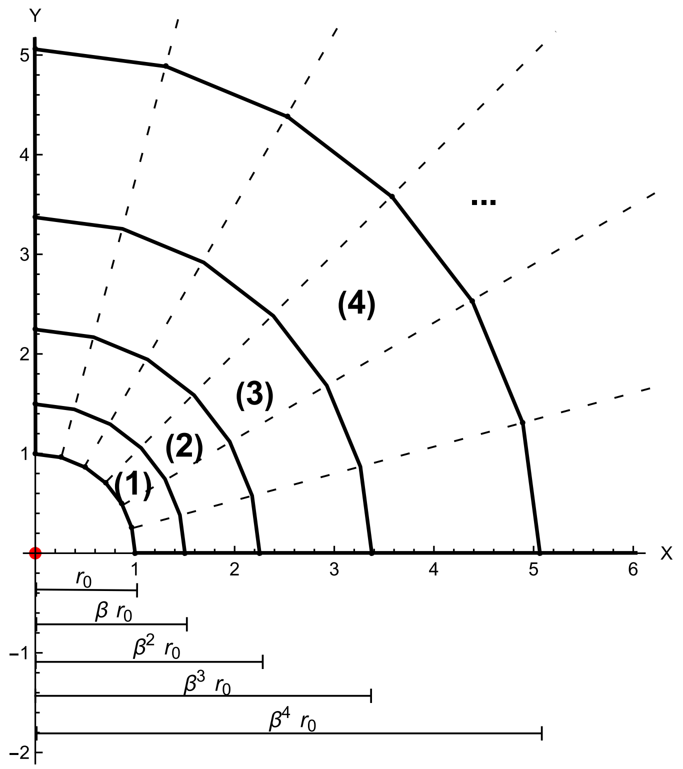


Figure 3: Proposed mesh of the infinite plate with aspect ratio $\beta = 1.5$

An initial compressive stress field $\sigma_{x0} = -1000$ Pa was applied. The stress distributions of the plate subjected to only the hole surface loading are shown in

Table 2. Stress concentration occurs primarily at the edge of the hole, diminishing to minor levels at a distance of one diameter from the edge, and becoming negligible at twice that distance.

After superposing the initial compressive state with the hole-induced transient stress, the principal stresses at the Gauss points along the approximating horizontal and vertical lines, which are at respective angles of 3.17° and 86.83°, are shown in Figure 4. This representation accurately illustrates the stress concentration factor, which approximates 3 in the vertical direction and rapidly decreases as the distance from the hole increases. The overall trends in the predictions of the proposed method compare favourably to the theoretical results, demonstrating that this simplified method facilitates an efficient and accurate representation of the static response in semi-infinite structural problems. There are oscillating deviations for the minimum principal stresses in the horizontal direction, a discrepancy that is attributed to the finite element approximation. Enhancements in accuracy can be realized through the use of higher-order elements or by refining the mesh within a typical layer.

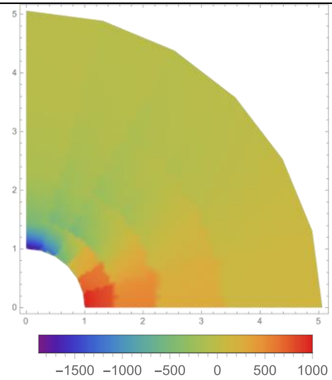
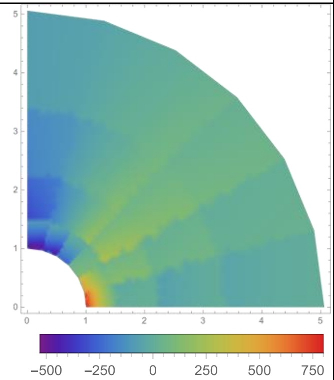
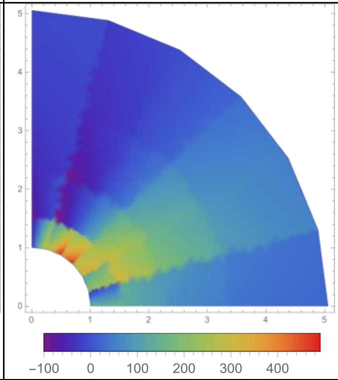
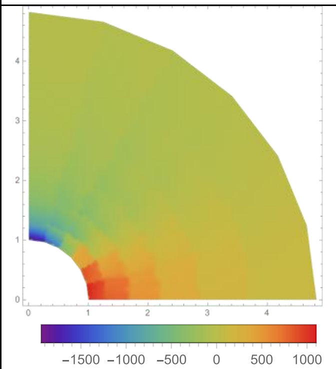
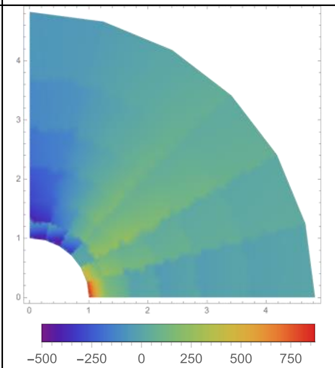
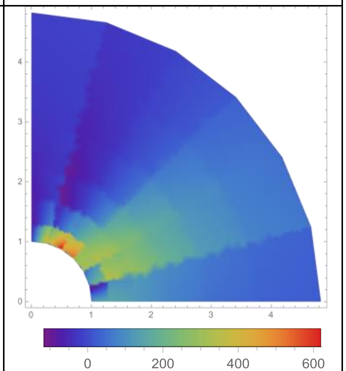
β	σ_x	σ_y	σ_{xy}
1.5			
	$\sigma_{x,\max} = 1017.78$ Pa $\sigma_{x,\min} = -1901.67$ Pa	$\sigma_{y,\max} = 806.21$ Pa $\sigma_{y,\min} = -559.40$ Pa	$\sigma_{xy,\max} = 538.70$ Pa $\sigma_{xy,\min} = -123.75$ Pa
1.3			
	$\sigma_{x,\max} = 1117.7863$ Pa $\sigma_{x,\min} = -1980.98$ Pa	$\sigma_{y,\max} = 903.91$ Pa $\sigma_{y,\min} = -583.69$ Pa	$\sigma_{xy,\max} = 671.46$ Pa $\sigma_{xy,\min} = -121.69$ Pa

Table 2: Hole-induced stress distributions

It is worth noting that the aspect ratio of the elements corresponds to an elongated rectangular shape with the geometric layer proportionality factor $\beta = 1.5$. A better choice of β can improve the results, since square elements generally produce more accurate results than rectangular elements in FE analysis. Therefore, the same analysis has been carried out with $\beta = 1.3$ for approximately square elements. The results of stress distributions and principal stress variation over space are shown in Table 2 and Figure 5, respectively. Evidently, with the same computational demand, the square element gives a smaller deviation and better resolution.

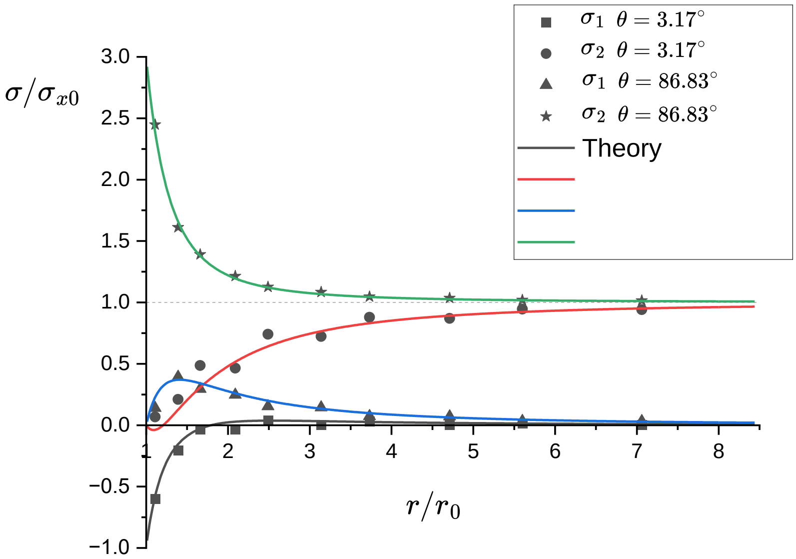


Figure 4: Comparison of principle stresses over space for $\beta = 1.5$

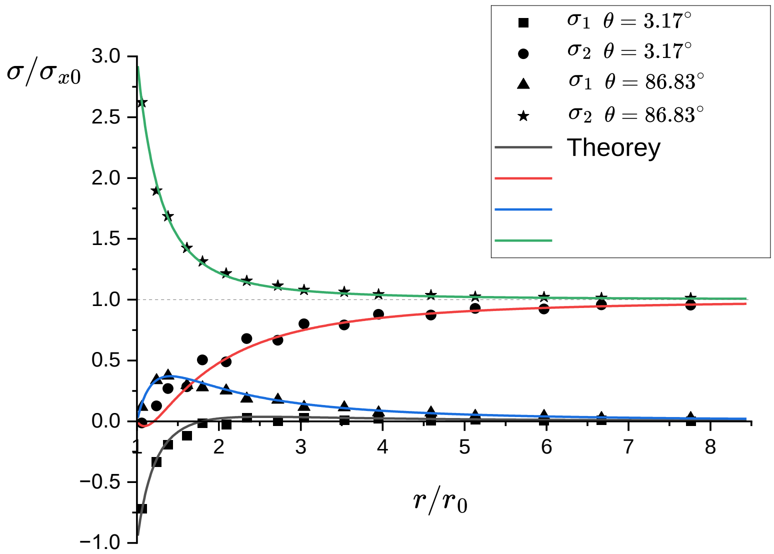


Figure 5: Comparison of principle stresses over space for $\beta = 1.3$

3.2 Soil foundation

This method is then applied to model a foundation problem with axis-symmetric elements. A uniform stress $\sigma_y = -1000$ kPa is applied to the surface of the circular foundation. In this case, the stiffness matrix increases linearly across each layer, with the proportionality factor α equalling the layer geometric ratio β. There is only one constant mode which represents the vertical rigid body movement of the mesh. The overall deformation is illustrated in Figure 6. As shown in Figure 7, the accuracy of the model is significantly improved through h-refinement, while the computational burden remains low since only a single layer of elements is analysed. This exemplifies the capability of the proposed method to substantially reduce computational effort, even when employing a highly refined mesh.

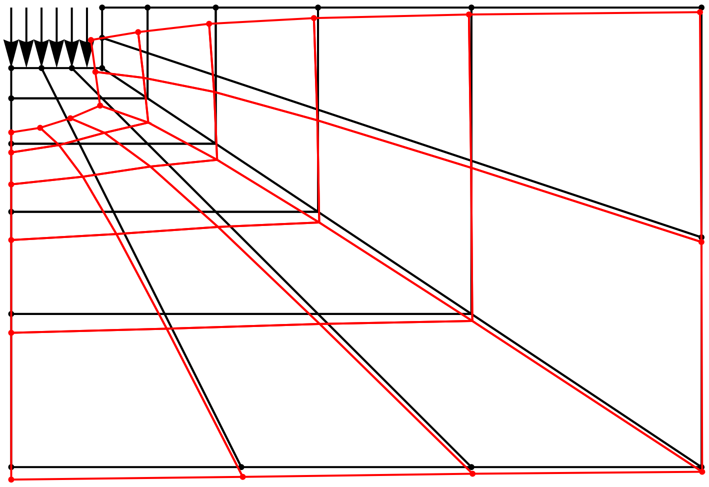
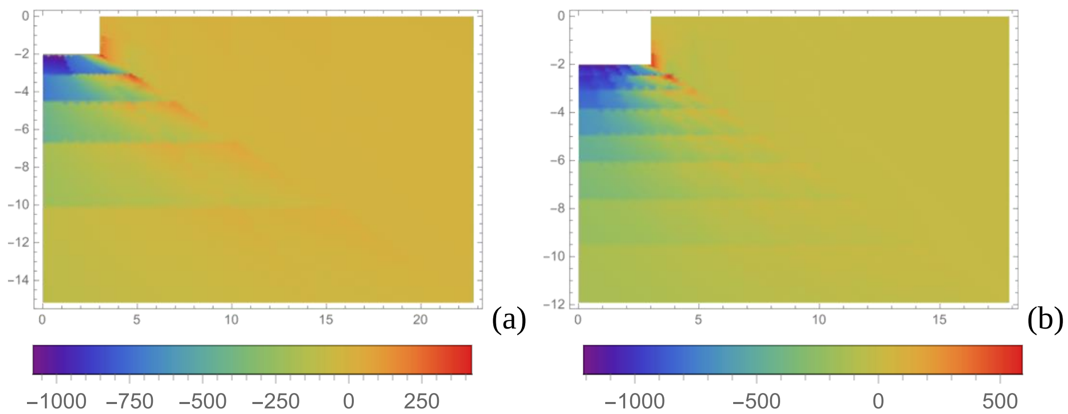


Figure 6: Deformed foundation under uniform vertical stress



$\sigma_{y,\max} = 481.28$ kPa $\qquad$ $\sigma_{y,\max} = 688.61$ kPa

$\sigma_{y,\min} = -1110.94$ kPa $\qquad$ $\sigma_{y,\min} = -1177.34$ kPa

Figure 7: Vertical stresses in soil with (a) coarse mesh; (b) fine mesh

4 Conclusions

This paper presents a finite element method tailored for simulating semi-infinite domain problems, leveraging structural periodicity to enhance computational efficiency. The domain is strategically divided into layers of proportionally dimensioned elements, ensuring that the stiffness matrix between layers remains constant or proportional. The method hinges on calculating the stiffness matrix of a single layer of elements to depict the static response of the entire domain, providing a computationally economical solution even when employing a fine mesh. It adeptly addresses the challenges posed by the extensive DOF typically associated with semi-infinite domains. Utilising eigenvalue analysis, the relationship between nodal deformation across different layers and mechanical behaviours including transient deformation, rigid body movement, and constant strains are examined. This facilitates the identification and extraction of recurrent modes involved in force transmission within the periodic structure, with the overall static response obtained using modal superposition. This method markedly reduces the need for additional elements and nodes in areas remote from the primary zone of interest, optimizing resource utilisation. However, it is predominantly applicable to linear problems involving homogeneous materials. The practical validity and advantages of the approach are demonstrated through a simulation of an infinite plate with a central hole under uniaxial loading. The resulting stress distributions closely align with theoretical predictions, demonstrating the efficacy of the method. Furthermore, the technique is applied to analyse a simple foundation problem and shows the computational superiority with a fine mesh. This promising approach can be extended to model other complex structural problems with a semi-infinite domain, including three-dimensional problems, such as soil-structure interaction and tunnelling, paving the way for broader applications in structural engineering.

References

[1] G. Beer and J. L. Meek,"' Infinite domain" elements', Int. J. Numer. Methods Eng., vol. 17, no. 1, pp. 43–52, 1981, doi: https://doi.org/10.1002/nme.1620170104.

[2] P. Bettess,"Infinite elements", Penshaw Press, 1992.

[3] M. H. Aliabadi, "The boundary element method. Applications in solids and structures", in "The boundary element method", Volume 2, Chichester: Wiley, 2002.

[4] D. J. Mead, "Wave propagation and natural modes in periodic systems: I. Mono-coupled systems", J. Sound Vib., vol. 40, no. 1, pp. 1–18, 1975, doi: https://doi.org/10.1016/S0022-460X(75)80227-6.

[5] E. Moses, M. Ryvkin, and M. B. Fuchs, "A FE methodology for the static analysis of infinite periodic structures under general loading", Comput. Mech., vol. 27, no. 5, pp. 369–377, May 2001, doi: 10.1007/s004660100249.

[6] L. Brillouin, "Wave propagation in periodic structures: Electric filters and crystal lattices", Second edition, New York: Dover Publications, 2003.

[7] G. Kirsch, "Die theorie der elastizität und die bedürfnisse der festigkeitslehre", Springer, 1898.