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Solving the Elasto-Plastic Behaviour of Two Bodies in Contact Using the Mortar Method

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Abstract

This paper focuses on the Mortar method employing a segment-to-segment approach, utilized for connecting non-conforming and non-overlapping meshes and for handling contact between two elasto-plastic bodies. We provide a brief overview of the theory, present our implementation in Matlab, and conduct numerical results. We use the vectorized aproach for assembling tangential stiffness matrices and the semi-smooth Newton method for linearizing the material nonlinearities.

Keywords: finite element method, elasto-plasticity, Mortar method, contact problem, semi-smooth Newton method, Matlab

1 Introduction

In civil engineering, particularly in structural analysis, designing joints between load-bearing members such as columns and beams is a critical aspect of structural design. In steel structures, welded joints are commonly utilized, where loads between members are transmitted via weld and bolt connections. Bolts handle shear and tension, while compression is typically transferred through contact pressure between steel plates. European standards, known as Eurocodes, offer comprehensive guidelines for joint design, including instructions and methods for calculating load-bearing capacity and rotation stiffness.

The advent of Finite Element Method (FEM) [1] software has led to a shift towards more universal models employing shell and beam elements. In many engineering applications, simulating contact pressure between bodies often involves introducing artificial rigid nonlinear thrust beams with additional boundary conditions to allow compression forces only. This method proves effective when bodies are initially in close proximity with minimal friction influence.

Another approach is the "Mortar method" [5] which facilitates connecting non-conforming meshes and enables contact between elasto-plastic bodies. This paper introduces the concept of Mortar methods for elasto-plastic contact problems in 2D. We present our Matlab implementation, utilizing fully vectorized Matlab code for elasto-plastic problems [10], and provide solutions for standard Hertz benchmark.

We consider a frictionless contact boundary condition between two bodies denoted as $\Omega^1,\ \Omega^2\ \subset \mathbb{R}^2$, see Figure 1. We assume that the bodies are fixed on the parts $\Gamma_U^1,\ \Gamma_U^2 \neq \emptyset$ of the boundaries. The load is represented by surface (prescribed on the boundaries parts $\Gamma_N^1,\ \Gamma_N^2$) and volume forces. The material of the bodies is described by the elasto-plastic constitutive model with the von Mises yield criterion and linear kinematic hardening [2]. For the sake of simplicity, we confine ourselves on one-step problem formulated in displacement. It leads to a minimization of the convex and smooth functional on a convex set. However the stress-strain relation is not smooth.

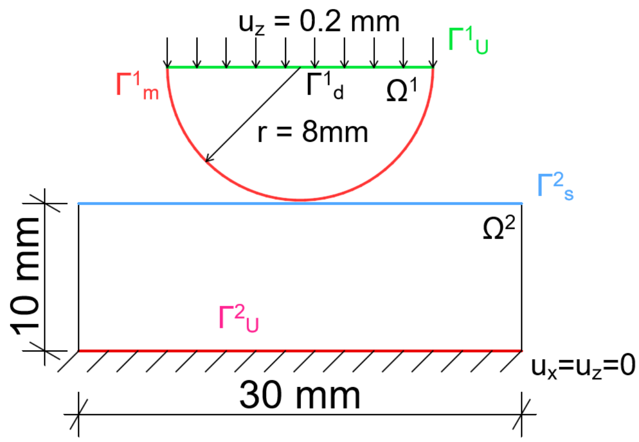

Figure 1: Hertz problem

For the numerical solution of the proposed problem, we adopt the commonly used FEM, see, e.g., [4], [3]. The finite element partition will be denoted as $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2$ and consists of elementary elements. In particular, displacement fields are approximated by continuous, piece-wise linear functions and strain (stress) fields are approximated by piece-wise constant functions. The final discretized problem can be classified as an optimization problem with simple linear equality and inequality constraints.

2 Algebraic formulation of contact problem for elasto-plastic bodies

The algebraic formulation of the problem is related to the contact of two bodies. This means that an unknown displacement column vector $\mathbf{v} \in \mathbb{R}^n$ consists of two parts, i.e., it has the following structure:

$$\mathbf{v} = \left(\mathbf{v}_1^T,\, \mathbf{v}_2^T\right)^T,$$

where $\mathbf{v}_i$ denotes the displacement vector on $\Omega^i, i = 1, 2$. We define the space

$$\mathcal{V} = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{B}_E \mathbf{v} = \mathbf{o}\}, \tag{1}$$

and the set of feasible displacements

$$\mathcal{K} = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{B}_E \mathbf{v} = \mathbf{c}_E,\ \mathbf{B}_I \mathbf{v} \leq \mathbf{c}_I\}. \tag{2}$$

Here the equality constraint matrix $\mathbf{B}_E \in \mathbb{R}^{m_E \times n}$ represents the Dirichlet boundary conditions defined on $\Gamma_U^1,\ \Gamma_U^2$. The inequality constraint matrix $\mathbf{B}_I \in \mathbb{R}^{m_I \times n}$ represents the non-penetration condition on the contact zones $\Gamma_N^1,\ \Gamma_N^2$. Notice that $\mathcal{K}$ is convex and closed.

Let $\mathbf{K}_e \in \mathbb{R}^{n \times n}$ be a block diagonal matrix consisting of the elastic stiffness matrices $\mathbf{K}_e^i$ defined on each domain $\Omega^i,\ i = 1, 2$. Due to the presence of the Dirichlet boundary conditions on both domains and the Korn inequality, we can define the energy norm on $\mathcal{V}$:

$$\|\mathbf{v}\|_e := \sqrt{\mathbf{v}^T \mathbf{K}_e \mathbf{v}} = \sqrt{\sum_{i=1}^{2} \mathbf{v}_i^T \mathbf{K}_e^i \mathbf{v}_i},\ \mathbf{v} = \left(\mathbf{v}_1^T,\, \mathbf{v}_2^T\right)^T \in \mathcal{V}.$$

Notice that using this norm is suitable both from mechanical and mathematical points of view.

The algebraic formulation of the contact elastic problem can be written as the following optimization problem

$$\text{Find } \mathbf{u} \in \mathcal{K} : J(\mathbf{u}) \leq J(\mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{K}, \tag{3}$$

where

$$J(\mathbf{v}) := \Psi(\mathbf{v}) - \mathbf{f}^T \mathbf{v}, \quad \mathbf{v} \in \mathbb{R}^n. \tag{4}$$

Here the vector $\mathbf{f} = \left(\mathbf{f}_1^T, \mathbf{f}_2^T\right)^T \in \mathbb{R}^n$ represents the load consisting of the volume and surface forces, and the initial stress state. The functional Ψ represents the inner energy and has the structure

$$\Psi(\mathbf{v}) = \left(\Psi_1(\mathbf{v}_1)^T, \Psi_2(\mathbf{v}_2)^T\right)^T.$$

Additionally, Ψ is a potential to the non-linear elasto-plastic operator $F : \mathbb{R}^n \to \mathbb{R}^n$, i.e., $D\Psi(\mathbf{v}) = F(\mathbf{v}), \forall \mathbf{v} \in \mathbb{R}^n$. The function F is generally nonsmooth but Lipschitz continuous. It enables us to define a generalized derivative $K : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ of F in the sense of Clark, i.e. $K(\mathbf{v}) \in \partial F(\mathbf{v}), \mathbf{v} \in \mathbb{R}^n$. Notice that $K(\mathbf{v})$ is symmetric, block diagonal and sparse matrix. Moreover the following properties of F and K hold [9]:

1.
$$F(\mathbf{v}+\mathbf{w}) - F(\mathbf{v}) = \int_0^1 K(\mathbf{v}+\theta\mathbf{w})\mathbf{w}\,\mathrm{d}\theta \quad \forall \mathbf{v},\mathbf{w} \in \mathbb{R}^n. \tag{5}$$

2. $K(\mathbf{v})$ is uniformly positive definite and bounded with respect to $\mathbf{v} \in \mathcal{V}$:
$$\exists \nu \in (0,1): \quad \nu\|\mathbf{w}\|_e^2 \leq \mathbf{w}^T K(\mathbf{v})\mathbf{w} \leq \|\mathbf{w}\|_e^2 \quad \forall \mathbf{v},\mathbf{w} \in \mathcal{V}. \tag{6}$$

3. F is strongly semismooth [6] on $\mathcal{V}$, which yields that for any $\mathbf{v} \in \mathcal{V}$ and any of sufficiently small $\mathbf{w} \in \mathcal{V}$:
$$F(\mathbf{v}+\mathbf{w}) - F(\mathbf{v}) - K(\mathbf{v}+\mathbf{w})\mathbf{w} = O(\|\mathbf{w}\|_e^2). \tag{7}$$

Notice that (5) and (6) yield that Ψ is coercive and strictly convex on $\mathcal{V}$. Hence the problem (4) has a unique solution and can be equivalently written as the following variational inequality:
$$\text{Find } \mathbf{u} \in \mathcal{K} \;:\; F(\mathbf{u})^T(\mathbf{v}-\mathbf{u}) \geq \mathbf{f}^T(\mathbf{v}-\mathbf{u}) \quad \forall \mathbf{v} \in \mathcal{K}. \tag{8}$$

The estimate (7) will be important for showing that the semi-smooth Newton method defined in the next section has a local quadratic convergence.

3 Semi-smooth Newton method for optimization problem

The proposed problem (3) contains two nonlinearities – the non-quadratic functional J (due to Ψ) and the non-penetration conditions included in the convex set $\mathcal{K}$. By the semismooth Newton method, we will approximate Ψ by a quadratic functional similar to the Taylor expansion:
$$\Psi(\mathbf{u}) \approx \Psi(\mathbf{u}^k) + F(\mathbf{u}^k)^T(\mathbf{u}-\mathbf{u}^k) + \frac{1}{2}(\mathbf{u}-\mathbf{u}^k)^T K(\mathbf{u}^k)(\mathbf{u}-\mathbf{u}^k),$$

for a given approximation $\mathbf{u}^k \in \mathcal{K}$ of the solution $\mathbf{u}$ to the problem (3). Let us denote $\mathbf{f}_k = \mathbf{f} - F(\mathbf{u}^k)$, $\mathbf{K}_k = K(\mathbf{u}^k)$ and define:
$$\mathcal{K}_k := \mathcal{K} - \mathbf{u}^k = \left\{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{B}_E\mathbf{v} = \mathbf{o},\ \mathbf{B}_I\mathbf{v} \leq \mathbf{c}_{I,k},\ \mathbf{c}_{I,k} := \mathbf{c}_I - \mathbf{B}_I\mathbf{u}^k\right\},$$
$$J_k(\mathbf{v}) := \frac{1}{2}\mathbf{v}^T\mathbf{K}_k\mathbf{v} - \mathbf{f}_k^T\mathbf{v}, \quad \mathbf{v} \in \mathcal{K}_k. \tag{9}$$

Then the Newton step is following:
$$\mathbf{u}^{k+1} = \mathbf{u}^k + \delta\mathbf{u}^k, \quad \mathbf{u}^{k+1} \in \mathcal{K},$$

where $\delta\mathbf{u}^k \in \mathcal{K}_k$ is a unique minimum of J_k on $\mathcal{K}_k$:
$$J_k(\delta\mathbf{u}^k) \leq J_k(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathcal{K}_k, \tag{10}$$

or equivalently $\delta\mathbf{u}^k \in \mathcal{K}_k$ solves the following inequality:

$$\left(\mathbf{K}_k\delta\mathbf{u}^k\right)^T(\mathbf{v}-\delta\mathbf{u}^k) \geq \mathbf{f}_k^T(\mathbf{v}-\delta\mathbf{u}^k) \quad \forall\mathbf{v}\in\mathcal{K}_k. \tag{11}$$

Notice that if we substitute $\mathbf{v} = \mathbf{u}^{k+1} \in \mathcal{K}$ into (8) and $\mathbf{v} = \mathbf{u} - \mathbf{u}^k \in \mathcal{K}_k$ into (11), then by adding we obtain the inequality

$$\left(K(\mathbf{u}^k)\delta\mathbf{u}^k\right)^T(\mathbf{u}-\mathbf{u}^{k+1}) \geq \left(F(\mathbf{u})-F(\mathbf{u}^k)\right)^T(\mathbf{u}-\mathbf{u}^{k+1}),$$

which can be arranged into the form

$$(\mathbf{u}^{k+1}-\mathbf{u})^T K(\mathbf{u}^k)(\mathbf{u}^{k+1}-\mathbf{u}) \leq \left(F(\mathbf{u}^k)-F(\mathbf{u})-K(\mathbf{u}^k)(\mathbf{u}^k-\mathbf{u})\right)^T(\mathbf{u}-\mathbf{u}^{k+1}).$$

Hence one can simply derive local quadratic convergence of the semi-smooth Newton method by (6) and (7) provided that $\mathbf{u}^k$ is sufficiently close to $\mathbf{u}$.

4 Mortar method

To describe contact between $\Gamma_N^1,\ \Gamma_N^2$, we divide contact zones into master-slave surfaces $\gamma_c^{(1)}$,$\gamma_c^{(2)}$ in such a way, that for every master surface $\gamma_c^{(1)}$ on Γ_N^1 there exists slave surface $\gamma_c^{(2)}$ on Γ_N^2. Between each master-slave contact surfaces, we can define discretized contact energy Π_c caused by traction forces $\mathbf{t} \in \mathbb{R}^{m_I}$ on gap $\mathbf{g} \in \mathbb{R}^{m_I}$ between those surfaces

$$\Pi_c(\mathbf{v},\mathbf{t}) = \mathbf{t}^T\mathbf{g} \tag{12}$$

with Karush–Kuhn–Tucker (KKT) conditions for friction-less contact

$$g_j \geq 0, \tag{13}$$

$$t_j \leq 0, \tag{14}$$

$$t_j g_j = 0. \tag{15}$$

Inequalities (13) ensure non-penetrating of bodies, inequalities (14) enforce the pressure exclusively on the interface, and complementarity equations (15) secure pressure to occur if and only if two bodies are in contact with zero gaps as well as prevent contact pressure to act in the case of a non-zero gap. Rewriting gap constrain (13) to the matrix representation, we obtain mortar inequality constrain

$$\mathbf{B}_I\mathbf{v} \leq \mathbf{c}_I \tag{16}$$

with mortar constrain matrix $\mathbf{B}_I$ and mortar constrain vector $\mathbf{c}_I$ given by

$$\mathbf{B}_I = \mathbf{N}^T(\mathbf{D}-\mathbf{M}), \tag{17}$$

$$\mathbf{c}_I = -\mathbf{N}^T(\mathbf{D}-\mathbf{M})\mathbf{x}, \tag{18}$$

where $\mathbf{x} \in \mathbb{R}^n$ is node coordinate vector and $\mathbf{N} \in \mathbb{R}^{2m_I \times m_I}$ is a block-diagonal matrix with diagonal blocks of the normal vector from each slave node $\mathbf{n}_i,\ i = \{1, \dots, sl\}$ on slave surface $\gamma_c^{(1)}$,

$$\mathbf{N} = \begin{bmatrix} \mathbf{n}_1 & & \\ & \ddots & \\ & & \mathbf{n}_{sl} \end{bmatrix}.$$

Matrices $\mathbf{M}, \mathbf{D} \in \mathbb{R}^{2m_I \times n}$ are composed from rows representing dependencies of the movement between master and slave nodes with components given by

$$\mathbf{D}_{j,j} = D_{j,j}\mathbf{I}_2 = \int_{\gamma_c^{(1)}} N_j^{(1)} d\gamma \mathbf{I}_2, \tag{19}$$

$$\mathbf{M}_{j,l} = M_{j,l}\mathbf{I}_2 = \int_{\gamma_c^{(1)}} \Phi_j N_l^{(1)} d\gamma \mathbf{I}_2. \tag{20}$$

Here, N_j and N_l denote basis functions of the slave and master mortar elements, respectively, Φ_j denotes dual basis functions of slave mortar element and $\mathbf{I}_2$ a 2×2 identity matrix. Both integrals are evaluated over the slave surface $\gamma_c^{(1)}$. The dimension of mortar elements is always one lower than the dimension of the elements (in our example we have 2D elements and 1D mortar elements) with the same basis functions as the normal elements. In our implementation, we consider linear elements with given explicit dual basis functions

$$\Phi_1 = 1/2(1 - 3\zeta), \qquad \Phi_2 = 1/2(1 + 3\zeta).$$

For higher order basis function, please refer to [5].

Additionally, we have to mention some technical details about mortar constraints. Each integration in (19), (20) is further multiplied by identity matrix $\mathbf{I}_2$ since there are two degrees of freedom (DOF) for each node. Further, normal vectors and node positions are based on the deformation of bodies, thus the inequality (16) is non-linear in $\mathbf{v}$, see our previous articles [7, 8]. The last equation $\mathbf{D} - \mathbf{M}$ describes the mutual movement of slave-master nodes respectively, which causes the sum of each row of $\mathbf{D} - \mathbf{M}$ to be zero. In special case, when all nodes from $\gamma_c^{(1)}$ lie on $\gamma_c^{(2)}$, the equation (18) would evaluate zero vector. In such a case, changing inequality in (16) to equality ensures the glue boundary condition of two non-conforming meshes.

5 Numerical benchmark

As a benchmark, we consider a contact between two bodies Ω^1, Ω^2 of homogeneous elasto-plastic material with zero displacement on boundary Γ_U^2 and imposed displacement $\mathbf{u_z}$ on Γ_U^1, see Figure 1. We consider the von Mises elasto-plastic model with kinematic hardening. Material of Ω^1 and Ω^2 are defined by Young's modulus $E_1 = E_2 = 206,900$ Pa and Poisson's ratio $\nu_1 = \nu_2 = 0.29$. The inelastic material parameters are set as follows: $a_1 = a_2 = 10,000$ Pa (kinematic hardening coefficient) and $Y_1 = Y_2 = 450\sqrt{2/3}$ Pa (yield stress). For this 2D benchmark, we consider plain strain. Body Ω^1 is a semicircle with radius of $r = 8$ mm and body Ω^2 is a rectangular with cross-section of $30/10$ mm. These blocks have prescribed contact zones Γ^1_m, Γ^2_s which also describe master and slave side of contact region respectively. Displacement is applied on face of the block Ω^1, spread equally across whole edge. Block Ω^1 is discretized by 1200 finite elements, block Ω^2 is discretized by 2720 finite elements, see Figure 2.

The termination criterion for the Newton method was chosen to be 10^{-6}, and the Matlab version of the Interior point method called "quadprog" was chosen as the interior solver of the problem with an inequality constraint.

In Figures 3 – 7, we can see the total displacement with the plotting of the total stress in the z-direction (Figures 4 and 5), also the von Mises stress (Figures 6 and 7) and in the Figures 3, you can see the value of hardening. All values are in Pa.

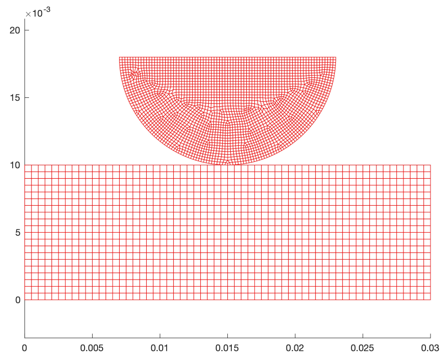

Figure 2: mesh of Hertz problem

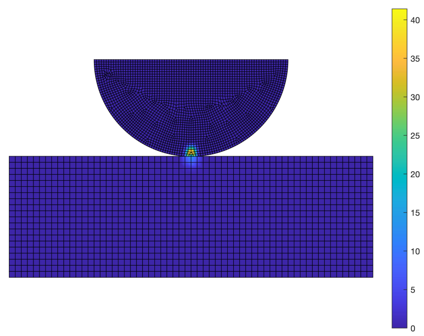

Figure 3: hardening fields

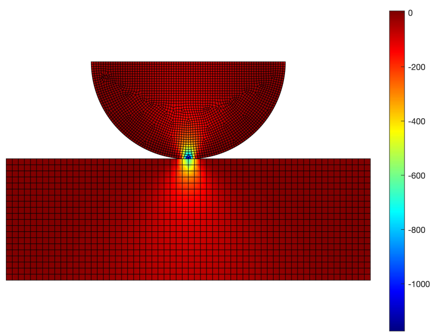

Figure 4: stress σ_y distribution

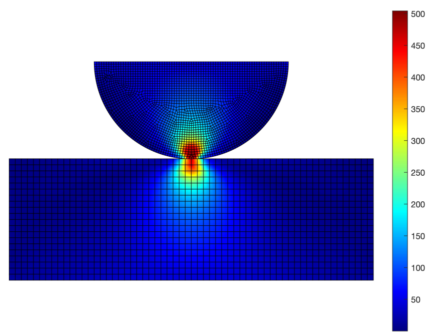

Figure 5: von Mises stress distribution

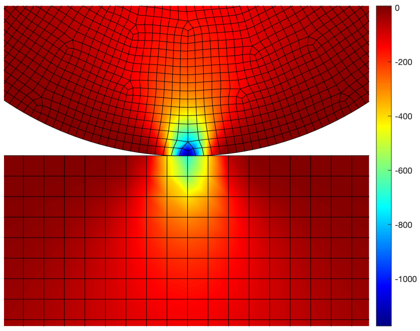

Figure 6: zoom of stress σ_y distribution

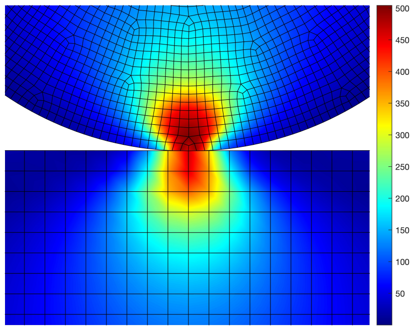

Figure 7: zoom of von Mises stress distribution

6 Concluding remarks

In this paper, we proposed a numerical method for solving contact elasto-plastic problems with the Mortar method on a numerical example. The numerical realization and implementation of the problem were newly included into the our in-house library. In fact, the proposed method can be used in other research or can be as a part of other contact inelastic problems than the considered frictionless contact problem of von Mises elasto-plastic bodies with kinematic hardening.

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