Introducing Anisotropic Eddy-viscosity Coefficient with Single-equation Model

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Abstract

A one-equation turbulence model is formulated which retains an anisotropic eddy viscosity coefficient. Consequently, the current model is deemed to be potential for accounting near-wall turbulence and strong flow in-homogeneity, enhancing the predictive accuracy for complex separated and reattaching flows. Furthermore, the devised turbulence model retrieves the link to the \( k-\varepsilon \) model and is likely to be extendable toward a non-linear algebraic Reynold stress model. Intuitively, the current artifact may accommodate a variable eddy-viscosity coefficient for an LES (large eddy simulation) or a DES (Detached eddy simulation) method.

Keywords: one-equation model, near-wall turbulence, non-equilibrium flow, LES, DES.

1 Introduction

To replicate both equilibrium and non-equilibrium flow in one-equation models, considerable innovative research has been undertaken [1–7]. Empiricism and arguments of dimensional analysis are involved in the widely used one-equation Spalart and Allmaras (SA) model [1], avoiding the link to the traditional \( k-\varepsilon \) turbulence model. Internal and external flows are extensively utilized to calibrate and validate the SA model; providing reasonable predictions. However, connection to the \( k-\varepsilon \) model has been ameliorated with recently developed one-equation models by Rahman et al. [3–6]. They reproduce relatively improved predictions for separated and reattaching flows owing to their capability in accounting for non-equilibrium
effects via variable model coefficients. In fact, a single-equation turbulence model establishes a good compromise between algebraic and two-equation models because of inheriting transport effects.

Customarily, one/two-equation models encounter non-equilibrium effects when being embedded with an anisotropic eddy-viscosity coefficient $C_{\mu}$, parameterized with a production to dissipation ratio $P_{k}/\varepsilon$ accompanied by invariants of mean strain-rate and vorticity tensors. The resulting $C_{\mu}$ suppresses non-physical energy components at moderate/severe strain rates on the perspective of realizability constraint, representing a minimal requirement for the turbulence model. Therefore, the current eddy-viscosity formulation with an appropriate strain-dependent $C_{\mu}$ reinforces turbulence anisotropy in a single-equation model. In addition, $k$ and $\varepsilon$ are explored in the present model, reviving presumably the competency in speculating complex separated and reattaching flows.

2 Formulation of present turbulence model

A transport equation for $R = C_{\mu} k^{2} / \varepsilon$ (pseudo-eddy viscosity) can be obtained using the two-equation $k$-$\varepsilon$ turbulence model. The following relation is used to construct an $R$-transport equation:

$$\frac{D R}{D t} = D\left(\frac{C_{\mu} k^{2} / \varepsilon}{D t}\right) = C_{\mu} \left(\frac{2k}{\varepsilon} \frac{D k}{D t} - \frac{k^{2}}{\varepsilon^{2}} \frac{D \varepsilon}{D t}\right) = \frac{2R}{k} \frac{D k}{D t} - \frac{R}{\varepsilon} \frac{D \varepsilon}{D t}$$

(1)

where the substantial derivative is indicated by $D/Dt$. Equations of $k$ and $\varepsilon$ at a high Reynolds number can be provided with:

$$\frac{D k}{D t} = \frac{P_{k} - \varepsilon}{\sigma_{k}} \nabla \cdot \left(\frac{R}{\sigma_{k}} \nabla k\right)$$

(2)

$$\frac{D \varepsilon}{D t} = C_{e1} \frac{\varepsilon}{k} P_{k} - C_{e2} \frac{\varepsilon^{2}}{k} + \nabla \cdot \left(\frac{R}{\sigma_{\varepsilon}} \nabla \varepsilon\right)$$

(3)

where $P_{k}$ implies the production term; relevant model constants are $\sigma_{k}, \sigma_{\varepsilon}, C_{e1}$ and $C_{e2}$. Combining Equations (1)-(3) and carrying out some algebra with $\sigma_{k} = \sigma_{\varepsilon} = \sigma_{R}$, result in an $R$-transport equation:

$$\frac{D R}{D t} = (2 - C_{e1}) \frac{R}{k} P_{k} - (2 - C_{e2}) k + \frac{\partial}{\partial x_{j}} \left[\frac{R}{\sigma_{R}} \frac{\partial R}{\partial x_{j}}\right] - \frac{2R^{2}}{k^{2} \sigma_{R}} \frac{\partial k}{\partial x_{j}} \frac{\partial k}{\partial x_{j}}$$

$$+ \frac{4R}{k} \frac{\partial k}{\partial x_{j}} \frac{\partial R}{\partial x_{j}} \frac{2}{\sigma_{R}} - \frac{\partial R}{\partial x_{j}} \frac{\partial R}{\partial x_{j}}$$

(4)

Apparently, the diffusion/destruction term $$\frac{4R}{k} \frac{\partial k}{\partial x_{j}} \frac{\partial R}{\partial x_{j}} - \frac{2R^{2}}{k^{2} \sigma_{R}} \frac{\partial k}{\partial x_{j}} \frac{\partial k}{\partial x_{j}}$$

appearing in the above-mentioned relation may be excluded in order to avoid the
numerical stiffness. Therefore, Equation (4) can be regularized using the Bradshaw relation \[|uv| = \sqrt{C_\mu k} = R \frac{du}{dy}\] [8] with the \(k-\varepsilon\) source and sink terms:

\[
\frac{DpR}{Dt} = C_1 \rho R S - C_\mu^* \rho k / 4 + \frac{\partial}{\partial x_j} \left[ \left( \mu + \frac{\mu_t}{\sigma_R} \right) \frac{\partial R}{\partial x_j} \right] = C_2 \rho \frac{\partial R}{\partial x_j} \frac{\partial R}{\partial x_j} \] (5)

where \(C_1 = \sqrt{C_\mu (C_{e2} - C_{e1})}\), \(C_{e1} = 1.44\), \(C_{e2} = 1.9 + C_\mu / 4\), \(C_\mu = 0.09\), \(C_2 = 2 / \sigma_R\), and \(C_{e1} = 1.44\), \(C_{e2} = 1.9 + C_\mu / 4\), \(C_\mu = 0.09\), \(C_2 = 2 / \sigma_R\) and \(\sigma_R = 1.3\)

The Park-Park limiter [9] is applied to determine the eddy-viscosity \(\mu_t\):

\[
\mu_t = \rho \min \left[ \frac{2 k T_i}{3 \varepsilon} ; \min \left( f_{vi} R ; C_\mu^* k T_i \right) \right] \] (6)

where the hybrid time scale \(T_i\) can be given by [5]:

\[
T_i = \sqrt{\frac{k^2}{\varepsilon^2} + C_T^2 \frac{v}{\varepsilon}} \frac{k}{1 + \frac{C_T^2}{\Re_T}} \] (7)

where \(\Re_T\) denotes the turbulent Reynolds number, \(C_T = \sqrt{2}\) is an empirical constant and \(v = \mu / \rho ; \rho\) is the density and \(\mu\) is the molecular viscosity signifies the kinematic viscosity.

The mean strain-rate \(S_{ij}\) and vorticity \(W_{ij}\) tensors, required afterward can be defined by

\[
S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad W_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \] (8)

The invariants of vorticity and mean strain-rate tensors can be represented by \(W = \sqrt{2 W_{ij} W_{ij}}\) and \(S = \sqrt{2 S_{ij} S_{ij}}\), respectively. The eddy-viscosity coefficient \(C_\mu^*\) in Equation (6) has been formed with mean strain-rate and vorticity invariants; \(C_\mu^*\) as suggested in Reference [5] has been adopted:

\[
C_\mu^* = \frac{\alpha_1}{1 + \frac{3}{2} \eta^2 + 2 \xi^2} \] (9)

where non-dimensional mean shear strain-rate and mean rotation-rate parameters are defined by \(\eta = \alpha_2 T_i S\) and \(\xi = \alpha_3 T_i W\), respectively. Coefficients \(\alpha_1-\alpha_3\) in Equation (9) are given by:

\[
\alpha_1 = g \left( \frac{1}{4} + \frac{2}{3} \sqrt{\Pi_b} \right), \quad \alpha_2 = \frac{3}{8} \sqrt{2} g, \quad \alpha_3 = \frac{3}{\sqrt{2}} \alpha_2, \quad g = \left( 1 + 2 \frac{P_k}{\varepsilon} \right)^{-1} \] (10)

with \(\Pi_b = b_{ij} b_{ij}\); the Reynolds stress anisotropy \(b_{ij}\) is characterized by
\[ b_y = \frac{u_r u_t}{2k} - \frac{1}{3} \delta_y \]  

Rahman et al. [3–6] have devised three explicit solutions to \( P_k / \epsilon \). The current formulation utilizes the simplest one, indicating a universal value of \( \sqrt{\Pi_b} \approx 0.31 \) in homogeneous turbulence. In addition, \( P_k / \epsilon \) receives an approximated consistency condition with \( P_k / \epsilon = \zeta / 3.2 \) in homogeneous turbulence, where \( \zeta = T, S \max (1, \Re \mu) \), and \( \Re = |W/S| \) indicates a non-dimensional variable [5].

The associated quantity \( f_{v1} \) in Equation (6) represents an eddy-damping function, defined by

\[ f_{v1} = \left[ 1 - \exp \left( -\frac{\nu^3}{L^2} \right) \right]^2, \quad L^2 = \zeta \left( 1 + C_{\mu}^* \Re \mu \right) \sqrt{\frac{
u^3}{L}} \]  

where \( \nu \) implies a wall-distance parameter usually normal to the wall and \( \left( \nu^3 / \epsilon \right)^{\nu/4} \) signifies the Kolmogorov length scale.

The tensor \( \bar{S} \) linked to the production term in Equation (5) represents the scalar measure of deformation; unlike the \( S4 \) turbulence model [1], this term is redefined in order to take the effect of vorticity into account:

\[ \bar{S} = f_x \left( S - \eta \right) \frac{\eta}{2}, \quad f_x = 1 - \frac{f_{v1}}{2} \sqrt{\max (1 - \Re \mu, 0)} \]  

where \( \eta = S - W \). Equation (13) has a close resemblance to an enhancement approach for the turbulence model sensitivity toward the effect of streamline curvature, provoking an extra rate of strain over and above the main strain-rate in the flow field. The kinetic energy of turbulence \( k \) can be approximated with the aid of Bradshaw’s relation as [11]:

\[ k = f_{v1}^3 \frac{R S_k}{C_{\mu}}, \quad S_k = \sqrt{S_a^2 + S_k^2} \]  

The non-vanishing strain-rate correction term \( S_a \) in the free-stream region can be designed using the log-law behaviour of pseudo-eddy viscosity \( R = u_t k y \) and \( du/dy = u_t / ky \) as:

\[ S_a = f_{v1} \max \left( \frac{3}{2k} \frac{\partial \sqrt{R}}{\partial \eta} \right)^2 \frac{1}{C_{\mu}^2} \]  

where \( 1/C_{\mu} s^{-1} \) has been estimated from a nearly homogeneous shear flow [10] and the von-Karman constant \( \kappa = 0.41 \). The hybrid time scale \( T_i \) requires an evaluation of the total dissipation-rate \( \epsilon \) since it plays an important role in constructing a compatible \( T_i \). \( \epsilon \) is determined as follows [11]:

\[ \epsilon = \sqrt{\epsilon^2 + \tilde{\epsilon}^2}, \quad \tilde{\epsilon} = R S_k f_{v1}^3 \]
where unlike $\varepsilon$, $\tilde{\varepsilon}$ disappears at the solid wall due to the product $\left(R \times f_{v1}^{1.3}\right)$. However, $\varepsilon_u$ indicates the wall-dissipation rate, balanced by the viscous-diffusion rate at the wall vicinity; $\varepsilon_u$ is conventionally modelled as:

$$\varepsilon_w = 2A_v \left(\frac{\partial u}{\partial y}\right)_w^2 \approx 2A_v S_k^2$$  \hspace{1cm}(17)$$

where $A_v = C_\mu = 0.09$ from DNS data. Apparently, the total dissipation-rate $\varepsilon$ is likely to be benefited by the wall dissipation-rate $\varepsilon_w$ within the wall-layer.

### 3 Results

Fully-developed turbulent channel flows at $Re_t = 180, 395$ and $640$ are simulated to substantiate the model accuracy in replicating near-wall turbulence. Computations are carried out in a half-width $h$ of the channel using a 1-D (one-dimensional) RANS solver. A non-uniform $1 \times 64$ grid resolution for $Re_t = (180; 395)$ and $1 \times 128$ grid resolution for $Re_t = 640$ are assumed to be adequate to accurately describe characteristics of the flow. To assure the viscous sublayer resolution, the first near-wall grid spacing is set to $y^+ \approx 0.3$. A cell-centered finite-volume approach is applied to solve the flow equations. Results are converted to the form of $u^+ = u/u'_r$, $k^+ = k/\nu r^2$, $\overline{uv}^+ = \overline{uv}/u'_r^2$, $\varepsilon^+ = v\varepsilon/\nu r^4$, where $u_r$ is the wall-friction velocity; comparisons are made by plotting these quantities versus $y^+ = y u_r / \nu$. Turbulence quantities are extracted from DNS data [12, 13]. Predictions of the present model are compared with those of the widely-used SA turbulence model [1].

The stream-wise mean $x$-momentum equation for a 1-D incompressible flow can be represented by

$$\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left[ (v + \nu_r) \frac{\partial u}{\partial y} \right] = 0$$ \hspace{1cm}(18)$$

where $\nu_r = \mu_r / \rho$ is the kinematic eddy-viscosity; lower and upper wall locations of the channel are encompassed by $y = (-h; h)$. The axial pressure-gradient $\partial p/\partial x$ remains constant and the continuity constraint $\partial u/\partial x = 0$ is naturally satisfied, as the mean flow field has a 1-D feature. However, $\partial p/\partial x$ must be computed as a part of the solution method since the pressure gradient is not known a priori. The pressure-velocity correction (PVC) method [14, 15] is an appropriate choice to solve the problem. The PVC scheme keeps updating the axial pressure gradient and velocity as long as the fictitious mass source is minimized.
Figure 1: Velocity profiles for fully-developed turbulent channel flow.

Figure 2: Shear stress profiles for fully-developed turbulent channel flow.

Figure 3: Kinetic energy profiles for fully-developed turbulent channel flow.

Figure 4: Dissipation-rate profiles for fully-developed turbulent channel flow.
Predicted profiles of the velocity and turbulent shear-stress from independent turbulence models are illustrated in Figures 1 and 2, respectively. It seems likely that indistinguishable predictive performances pertaining to both models are obtained. As can be seen, two turbulence models make pretty good correspondence with DNS data in both regions, comprising the linear boundary layer and wake defect layer. Present model performances are further assessed with turbulent kinetic energy $k^+$ and dissipation-rate $\varepsilon^+$ profiles as shown in Figures 3 and 4, respectively. Note-worthily, reasonable agreement of the current model with DNS data is visible without having transport and diffusion effects of the turbulent kinetic energy and dissipation-rate. It appears that the near-wall $k^+$-profile is qualitatively well reproduced and the maximum magnitude of $\varepsilon^+$ is captured in the wall-vicinity, as dictated by DNS and experimental data.

4 Conclusions

A compatible eddy-viscosity coefficient is introduced with the current model, the potential importance of which is not obvious since only a fully-developed turbulent channel flow case (e.g., simple shear flow case) is computed for validation. However, it is believed that the modification is profoundly convenient to account for strong flow in-homogeneity and near-wall turbulence and therefore, it can enhance the model competency in speculating complex separated and reattaching flows to a greater extent. Articulately, the link to the $k-\varepsilon$ model with the present model is retrieved and likely to be extendable toward a non-linear algebraic Reynolds stress model. Intuitively, the present formulation may accommodate a variable eddy-viscosity coefficient for an LES or a DES method.

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